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Updated QCD Analysis of the Bjorken Sum Rule Based on the Analytic Perturbation Theory

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We extend our theoretical study of the Bjorken sum rule for polarized deep inelastic scattering by building on recent Jefferson Lab data for the nucleon spin structure in the infrared region of small transferred momenta $Q^2 < 1$ GeV^2 where the higher order perturbative corrections and higher-twist contributions start play a significant role. Our theoretical approach is based on the analytic perturbation theory and an original method for matching regions with large and low Q^2 scales. Recall that in the infrared region, the application of conventional perturbation theory is problematic because the running coupling $\alpha_s(Q^2)$ has unphysical singularities, such as a ghost pole at $Q^2 = \Lambda^2$ in one-loop, and higher orders corrections do not eliminate this problem. Having fixed the QCD parameters at large Q^2, we move to the region of small Q^2, using the Gerasimov–Drell–Hearn sum rule as a boundary condition, thereby avoiding the problem of divergence at $Q^2 \to 0$ of a higher-twist representation as an expansion in inverse powers of $1/Q^2$. We investigate the stability of our theoretical predictions and the agreement between the theoretical description and the experimental data at Q^2 values very close to zero.

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1. Introduction

The deep inelastic scattering (DIS) of leptons on nucleons is one of the key processes in studying the internal structure of the of nucleons. The cross sections of this process are described by structure functions, the integrals of various combinations of which form the sum rules for the DIS structure functions. The rich experimental material accumulated at present on the first moments of the proton and neutron structure functions [1–14], as well as on the Bjorken sum rule [15], opens up the possibility of testing theoretical predictions and obtaining more accurate information on the higher-twist series which are more visible at low Q^2 scales, where a theoretical analysis includes both perturbative and higher-twist corrections related to each other. As is well known, the QCD description based on the use of ordinary perturbation theory (PT) is ill-defined at low energies scales, since the QCD running coupling $\alpha_s(Q^2)$ is not small and has unphysical singularities such as a ghost pole, which are in conflict with the fundamental principle of causality. Here, as before, we apply the analytic QCD approach proposed in [16] (see [17] for more details) called Analytic Perturbation Theory (APT). This approach takes into account the basic principles of local quantum field theory [18], which in the simplest cases is reflected in the form of Q^2-analyticity of the Källén–Lehmann type. The APT has been applied to the Bjorken sum rule in numerous papers including [19–27]. We continue the investigations of this sum rule within the ATB [28] framework, focusing on the region of small transferred momenta $Q^2 \leq 1$ GeV^2, including a very small momentum transfers, $Q^2 \leq 0.1$ GeV^2. As shown earlier in Ref. [19], using APT and taking into account the HT terms, a good description of the low-energy data from Jefferson Lab (JLab) can be achieved up to $Q_{min} \sim \Lambda_{\rm QCD} \sim 300$ MeV. In the current work, we are studying the possibility of describing JLab data for momentum transfers close to zero [29]. Since the nonperturbative part in the Bjorken sum rule has a series of powers of $1/Q^2$ (higher twists terms) which should move near $Q^2 \to 0$ into a function unknown so far, we use the technique of matching the function for large Q^2 with the behavior at the lowest Q^2, using for this purpose the Gerasimov–Drell–Hearn sum rule [30], see also [31, 32], as a boundary condition.

2. Theoretical review

Among DIS sum rules, the polarized Bjorken sum rule, Γ_1^{p-n}, is central to the study of the nucleon spin structure [15]. This sum rule is determined by the integral of the difference between the spin-dependent structural functions $g_1(x, Q^2)$ of the proton g_1^p and the neutron g_1^n over all possible values of the Bjorken variable x for a fixed square of the transferred momentum Q^2. In the limit $Q^2 \to \infty$, the expression for the Bjorken sum rule obtained using the algebra of currents and isospin symmetry has the form

$$\Gamma_1^{p-n}(Q^2)|_{Q^2\to\infty} = \int\limits_0^1 \left[g_1^p(x, Q^2) - g_1^n(x, Q^2)\right] dx = \frac{g_A}{6}\,, \tag{1}$$

where g_A is the axial coupling constant measured in neutron β decay.

Away from the large Q^2 limit, the Eq. (1) can be generalized for finite Q^2 and according to the operator product expansion (OPE) [33] is represented as

$$\Gamma_1^{p-n}(Q^2) = \frac{g_A}{6}\left[1 - D_{\rm BS}(Q^2)\right] + \sum_{i=2}^{\infty} \frac{\mu_{2i}(Q^2)}{Q^{2i-2}}\,, \tag{2}$$

where $D_{\rm BS}(Q^2)$ is the perturbation correction and μ_{2i}/Q^{2i-2} are the higher-twist (HT) contributions.

The perturbative QCD correction $D_{\rm BS}(Q^2)$ has a form of power series in QCD running coupling $\alpha_s(Q^2)$. At the up-to-date four-loop (N^{3}LO) level in the massless case it looks

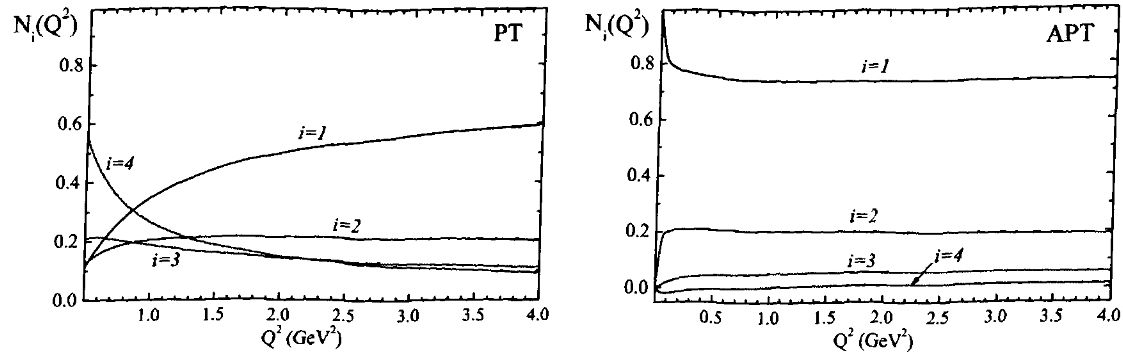

Figure 1: The Q^2-dependence of relative contributions of perturbative terms in PT (the left panel) and APT (the right panel), Eqs. (3) and (4), at the four-loop level (N^{3}LO).

like

$$D_{\rm BS}(\alpha_s) = \sum_{k\leq 4} c_k \alpha_s^k = 0,318\,\alpha_s + 0,363\,\alpha_s^2 + 0,652\,\alpha_s^3 + 1,804\,\alpha_s^4\,. \tag{3}$$

Here, the numerical expansion coefficients c_i in the modified minimal subtraction ($\overline{\rm MS}$) scheme, for three active flavors, $n_f = 3$, read as: $c_1 = 1/\pi = 0.31831$, $c_2 = 0.36307$, $c_3 = 0.65197$ and $c_4 = 1.8042$ [34].

In the framework of APT the α_s-series in (3) is replaced by expansions over analytic coupling functions:

$$D_{\rm BS}^{\rm PT} = \sum_{k\leq 4} c_k \alpha_s^k \quad \Rightarrow \quad D_{\rm BS}^{\rm APT} = \sum_{k\leq 4} c_k\, \mathcal{A}_k;$$

$$D_{\rm BS}^{\rm APT}(Q^2) = c_1\mathcal{A}^{(1)}(Q^2) + c_2\mathcal{A}^{(2)}(Q^2) + c_3\mathcal{A}^{(3)}(Q^2) + c_4\mathcal{A}^{(4)}(Q^2)\,, \tag{4}$$

where the coefficients c_1, c_2, c_3, and c_4 are the same as in (3), and the functions $\mathcal{A}^{(k)}(Q^2)$ are expressed in terms of the spectral functions $\varrho_k(\sigma) = {\rm Im}\,\left[a_{\rm PT}^k(-\sigma - i\epsilon)\right]$, using the Källén–Lehmann type representation

$$\mathcal{A}^{(k)}(Q^2) = \frac{1}{\pi}\int_0^\infty d\sigma\, \frac{\varrho_k(\sigma)}{\sigma + Q^2}\,. \tag{5}$$

Note that the first function in (5), i.e. $k = 1$, is associated with the analytic running coupling $\alpha_{\rm APT}(Q^2) = \pi\mathcal{A}^{(1)}(Q^2)$.

The difference in the convergence properties of PT and APT expansions, Eqs. (3) and (4), respectively, is demonstrated in Fig. 1. The value of the relative contribution of the term, $N_i(Q^2)$, is determined as

$$N_i^{\rm PT}(Q^2) = c_i\alpha_s^i(Q^2)/D_{\rm BS}^{\rm PT}(Q^2) \quad \text{and} \quad N_i^{\rm APT}(Q^2) = c_i\mathcal{A}^{(k)}(Q^2)/D_{\rm BS}^{\rm APT}(Q^2). \tag{6}$$

Let us turn to discussion of the HT contribution in (2). For very small values of Q^2, the HT representation consists of an infinite number of series terms that must sum to an as yet unknown function. To avoid this problem, the so-called "massive" representation in the form of twist-4 [35], that includes part of the contributions of the higher twist, is used,

$$\sum_{i=2}^{\infty} \frac{\mu_{2i}(Q^2)}{Q^{2i-2}} \to \frac{\hat{\mu}_4}{Q^2 + m_{ht}^2}.$$

Then, expression (2) takes the form

$$\Gamma_1^{p-n}(Q^2) = \frac{g_A}{6}\left[1 - D_{\rm BS}(Q^2)\right] + \frac{\hat{\mu}_4}{Q^2 + m_{\rm HT}^2}. \tag{7}$$

We will use the values of $\hat{\mu}_4$ and $m_{\rm HT}^2$ from the paper [36], see also [21], equal to $m_{\rm HT} = 0.439 \pm 0.012$ and $\hat{\mu}_4 = -0.082 \pm 0.002$.

3. Matching procedure

For the purpose of a smooth continuation of $\Gamma_1^{p-n}(Q^2)$ to the non-perturbative region, we follow the approach proposed in [37] and consider the integral

$$I_1(Q^2) \equiv \frac{2M^2}{Q^2}\Gamma_1(Q^2) = \frac{2M^2}{Q^2}\int_0^1 g_1(x, Q^2)dx, \tag{8}$$

where M is the mass of the nucleon. According to the Gerasimov–Drell–Hearn sum rule [30], the value of this integral at $Q^2 = 0$ is

$$I_1(0) = -\frac{\kappa_N^2}{4}, \tag{9}$$

where κ_N is the nucleon anomalous magnetic moment.

Taking into account (9) for the difference of moments corresponding to the Bjorken sum rule, we obtain

$$I_{\rm GDH}^{p-n} = \lim_{Q^2\to 0}\left[I_1^p(Q^2) - I_1^n(Q^2)\right] = \\ = -\frac{1}{4}\left(\kappa_p^2 - \kappa_n^2\right) \simeq 0.112, \tag{10}$$

as $\kappa_p = 1.793$ and $\kappa_n = -1.913$. If $Q^2 \to \infty$, then the Q^2-evolution of the integral $I_1^{p-n}(Q^2) = I_1^p(Q^2) - I_1^n(Q^2)$ looks like this:

$$\lim_{Q^2\to\infty} I_1^{p-n}(Q^2) = \frac{M^2}{Q^2}\frac{g_A}{3}. \tag{11}$$

Since $I_1^{p-n}(Q^2)$ has the same (positive) sign for large and small Q^2, is possible to obtain its smooth interpolation between large Q^2 and $Q^2 = 0$. Following [37], we decompose $I_1^{p-n}(Q^2)$ into a power series at some point Q_0 and define the expression at low Q^2 as:

$$I_1^{p-n}(Q^2) = \theta(Q^2 > Q_0^2)I_1(Q^2) + \theta(Q^2 < Q_0^2)\sum_{l=0}^{k}\frac{1}{l!}\frac{\partial^l I_1(Q^2)}{(\partial Q^2)^l}\bigg|_{Q=Q_0}(Q^2 - Q_0^2)^l. \tag{12}$$

Here k is the number of continuous derivatives of these two expansions, which turns out to be a free parameter of the model along with the value of the matching point Q_0. They must be chosen in such a way that the condition is satisfied:

$$I_1^{p-n}(0) = I_{\rm GDH}^{p-n}. \tag{13}$$

Figure 2 shows the results of matching depending on the order of the perturbative expansion (LO, NLO, N^{2}LO and N^{3}LO) and on the number of terms in the expansion (12) (the numbers on the curves). As can be seen from this figure, the behavior of $I_1^{p-n}(Q^2)$ changes only slightly with the change of the perturbative order, which is typical when using APT.

Table 1 shows the dependence of the matching point, Q_0, on the used order, k, for the perturbative part. As can be seen from this table, the point Q_0 varies slightly shifting from an order to a higher order to the region of large values Q^2. Figure 2 also demonstrates the same.

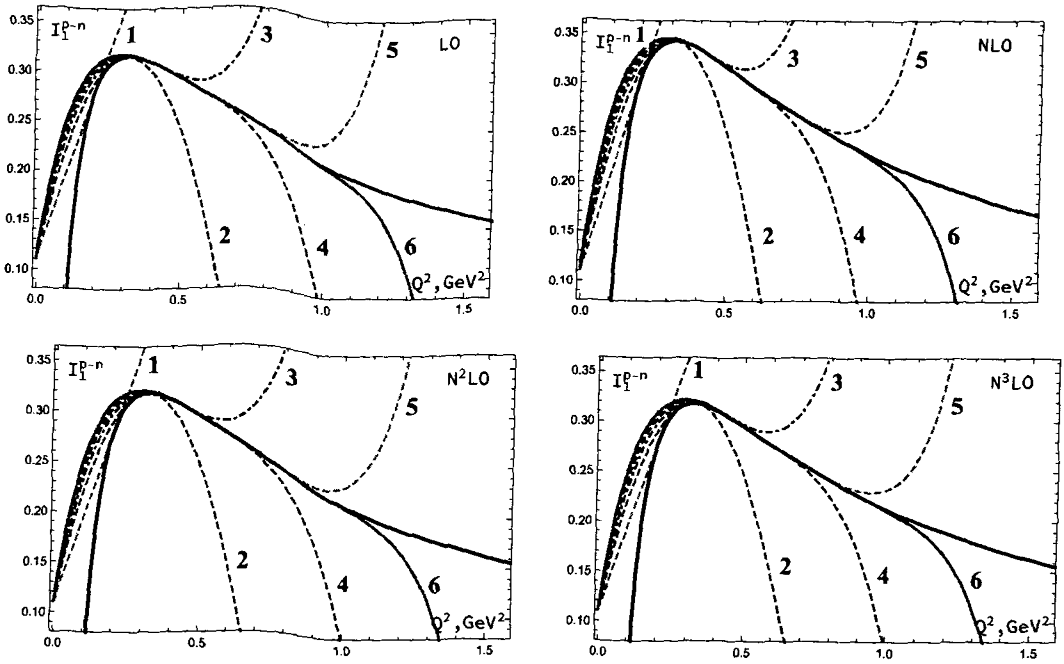

Figure 2: Behavior of $I_1^{p-n}(Q^2)$, Eq. (12), depending on the perturbative order (LO, NLO, N^{2}LO and N^{3}LO) and on the number of terms in the sum (numbers near the curves).

	LO	NLO	N^{2}LO	N^{3}LO
k = 1	0.199	0.196	0.203	0.205
k = 2	0.284	0.280	0.292	0.292
k = 3	0.369	0.363	0.378	0.379
k = 4	0.456	0.448	0.476	0.467
k = 5	0.542	0.543	0.554	0.555

Table 1: Matching points Q_0^2 (in GeV^2) for different orders of APT (LO, NLO, N^{2}LO and N^{3}LO) and the number k terms in the sum (12).

4. Fitting result

The result of fitting the JLab experimental data for $I_1^{p-n}(Q^2)$ in the region $Q^2 \leq 5$ GeV^2 is shown in Fig. 3, where the behavior within the framework of PT with matching is shown by a solid line, the PT without matching by a dashed line, and within the framework of the APT approach with matching, see Eqs. (7) and (12), by a dotted line. Figure 3 demonstrates that in the region of a few GeV^2, all theoretical curves are in good agreement with the experimental data, according to which we fixed the scale parameter $\Lambda_{\rm QCD}$. As for the region below $Q^2 < 2$ GeV^2, the behavior of the curves is different. This difference is shown in more detail in Fig. 4. As can be seen from this figure, which additionally shows the APT behavior without matching (short dashed line), for $Q^2 < 1$ GeV^2, the result of APT is in good agreement with the JLab data, whereas within the framework of the usual PT with matching, and especially without matching, there is no agreement for values of Q^2 close to zero.

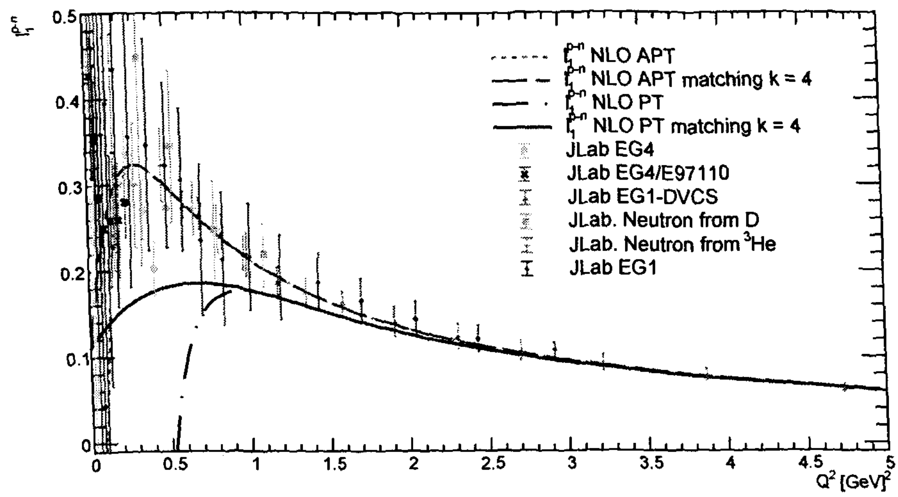

Figure 3: The result of fitting $I_1^{p-n}(Q^2)$ in various approach: the solid line corresponds to the PT with matching, the dot-dashed line the PT without matching, the dashed line to the APT approach with matching, Eq. (12).

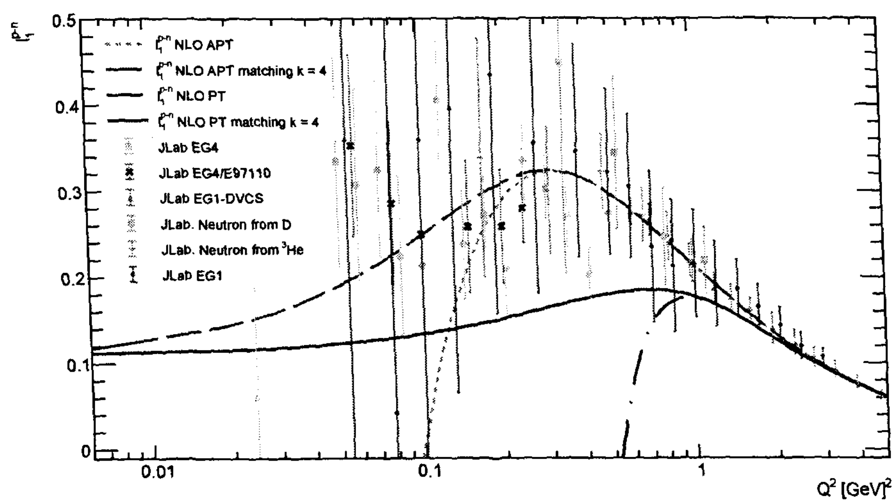

Figure 4: The behavior $I_1^{p-n}(Q^2)$ in various approach in the region of small transferred momenta. The solid line corresponds to the PT with matching, the dot-dashed line the PT without matching, the dashed line to the APT approach with matching and the short dashed line to the APT without matching.

5. Conclusion

In the present work we continued the study of the Bjorken sum rule in the region of small transferred momenta. We used a theoretical approach based on analytic perturbation theory (APT) and an original method of matching regions with large and small Q^2 scales. After the parameter $\Lambda_{\rm QCD}$ was fixed at large Q^2, we proceeded to describe the experimental JLab data at small Q^2 using the Gerasimov–Drell–Hearn sum rule as a boundary condition. We have demonstrated a acceptable theoretical description of JLab data for the Bjorken sum rule within the constructed approach down to record low values of Q^2, very close to zero. It would be highly desirable to have more accurate experimental data in this area.

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