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On relativization of the Gamov–Sommerfeld–Sakharov factor

Yu. D. Chernichenko, O. P. Solovtsova
International Center for Advanced Studies,
Gomel State Technical University, 48 October av., Gomel, 246746, Belarus

The new method of the relativization of S-factor Gamov–Sommerfeld–Sakharov in quantum chromodynamics was executed. Consideration is conducted within the framework of quasipotential approach in quantum field theory formulated in the relativistic configurational representation in the case of two particles of unequal masses.

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1. Introduction

By considering the total cross section for the production of a fermion-antifermion (or a quark-antiquark) pairs in e^+e^- annihilation in the kinematic region close to the threshold we can not cut off the perturbative series in powers of the fine structure constant α (i.e., in the number of loops), even if the expansion parameter α is small [1]. These state interactions lead to a large enhancement of the cross section if the particles are subject to a strong attractive interaction; they can lead to a large suppression under a strong repulsive interaction. The problem is well-known from QED [2]. This can be seen by considering the contributions of the magnetic, G_m, and electric, G_e, form factors to the total cross section for the production of a fermion-antifermion (or a quark-antiquark) pairs in e^+e^- annihilation in the kinematic region close to the threshold $s = 4m^2$ (we use the system of units $c = \hbar = 1$) [3]:

$$\frac{\sigma(e^+e^- \to f\bar{f})}{\sigma_{pt}} = v\left[|G_m|^2 + \frac{1}{2}\left(1-v^2\right)|G_e|^2\right], \quad \sigma_{pt} = \frac{4\pi\alpha^2}{3s}, \tag{1}$$

where $v = \sqrt{1-4m^2/s}$ is the relative velocity of fermions (or quarks) in the c.m. frame above threshold, s being the total c.m. energy of interacting particles and m is them mass.

The leading term, $\alpha\pi/2v$, of the expansion in the number of loops (i.e., in the powers of the fine structure constant α) of the moduli squared of the magnetic and electric form factors close to the threshold having the form [3]

$$\begin{aligned} |G_m|^2\big|_{v\to 0} &= 1 + \frac{\alpha\pi}{2v} - \frac{4\alpha}{\pi} + \frac{\alpha\pi\, v}{2} - \frac{\alpha}{3\pi}\left[4\ln\left(\frac{m^2}{\lambda^2}\right) - \frac{1}{3}\right]v^2 + O(v^3),\\ |G_e|^2\big|_{v\to 0} &= 1 + \frac{\alpha\pi}{2v} - \frac{4\alpha}{\pi} + \frac{\alpha\pi\, v}{2} - \frac{\alpha}{3\pi}\left[4\ln\left(\frac{m^2}{\lambda^2}\right) + \frac{8}{3}\right]v^2 + O(v^3), \end{aligned} \tag{2}$$

is the familiar Coulomb singularity [3, 4] which represents a long-distance effect and diverges for $v \to 0$. In expressions (2) also contains the well-known soft photon divergence $\propto \ln(m^2/\lambda^2)$ with a fictitious small photon mass $\lambda\,(\lambda/m \ll v \ll 1)$ which arises from the masslessness of the photon. This divergence have the order v^2 and would cancel with the corresponding soft photon divergence coming from the process of real radiation of one photon off one of the fermions according to the Kinoshita–Lee–Nauenberg theorem [5, 6]. The next term to leading contribution in the velocity expansion in Eqs. (2) is $-4\alpha/\pi$. It contribution represents a short-distance correction (of order $1/m$) to the production of a fermion-antifermion pair (the region of factorize). Therefore, the leading contribution in the velocity expansion is obtained by a resummation of Feynman diagrams with any number of exechanged photons where a diagram with a large number of loops corresponds to a large number of exechanged photons is needed to arrive at a sensible description of the interaction between the fermion-antifermion (or a quark-antiquark) pairs. In the nonrelativistic of case for the Coulomb interaction

$$V(r) = -\frac{\alpha}{r} \tag{3}$$

these threshold singularities in the form $(\alpha/v)^n$ can be explicitly summarized by the known S-factor Gamov–Sommerfeld–Sakharov [7–9]

$$S_{\rm nr} = \frac{X_{\rm nr}}{1-\exp(-X_{\rm nr})}, \qquad X_{\rm nr} = \frac{\pi\alpha}{v_{\rm nr}}, \tag{4}$$

which is related to the normalizated wave function of the continuous spectrum at the origin by $|\psi(0)|^2$. Here $2v_{\rm nr}$ is the relative velocity of two nonrelativistic particles. The corresponding nonrelativistic expression can also be obtained for higher ℓ states (see, e.g., [10]).

The problem of the final-state Coulomb interaction between the components of the electron-positron pair was first raised within the noncovariant perturbation theory based on the nonrelativistic Schrödinger equation in the well-known work by Sakharov [9]. In this work the interaction that brings to the production (annihilation) of pair, acts on distances $\lesssim 1/m$ while an essential distances for the Coulomb interaction are $\sim 1/m\alpha \gg 1/m$. This approach leades to the nonrelativistic S-factor in (4). Another solution of the final-state (or initial-state) Coulomb interaction problem for an arbitrary process involving charged particles with small relative velocities (i.e., nonrelativistic particles for which $\alpha/v \gtrsim 1$) was obtained in [11] by summing perturbation theory diagrams of the leading order in α/v.

In spite of the fact that S-factor in form is the same for different approaches and spins, in the relativistic theory the nonrelativistic approximation needs to be modified. For the first time the relativization of the S-factor (4) in QCD in the case of two particles of equal masses ($m_1 = m_2 = m$) was executed in [12, 13] and it consisted in the change $v_{\rm nr} \to v$. This factor was used for the description of effects close to the threshold of pair production in the processes $e^+e^- \to t\bar{t}$ and $e^+e^- \to W^+W^-$. They have shown that the instability of W bosons would not change the Coulomb enhancement factor of the $e^+e^- \to W^+W^-$ total cross section. Just same form of the S-factor but with the change $v_{\rm nr} \to v_{\rm rel} = 2v/(1+v^2)$ for the interaction of two particles of equal masses was later suggested in [3]. Another form of the relativistic generalization of the S-factor also in the case of two particles of equal masses was obtained in [14]. The relativistic S-factor for two particles of arbitrary masses ($m_1 \neq m_2$) was presented in [15]. This factor was derived within the framework of relativistic quantum mechanics on the basis of the Schrödinger equation.

The new method to the relativization of the S-factor in the case of two particles of equal masses was developed by Milton and Solovtsov in [16]. Their the method is based on the relativistic quasipotential (RQP) approach proposed by Logunov and Tavkhelidze [17] in the form suggested by Kadyshevsky [18]. This approach is a new step to application of the quasipotential approach in QCD and it gives the following expression for the relativistic S-factor:

$$S(\chi) = \frac{X(\chi)}{1-\exp\left[-X(\chi)\right]}, \qquad X(\chi) = \frac{\pi\alpha}{\sinh\chi}, \tag{5}$$

where χ is the rapidity related to the total c.m. energy of interacting particles, $\sqrt{s}$, by $2m\cosh\chi = \sqrt{s}$. The function $X(\chi)$ in Eq. (5) can be expressed in terms of v as $X(\chi) = \pi\alpha\sqrt{1-v^2}/v$.

We note that in the method developed by them, the possibility of transformation of quasipotential (QP) equation from momentum space into relativistic configurational representation in the case of two particles of equal masses (see [19]) has been used also. Moreover, it is important the potential (3) that used by them possesses the QCD-like behaviour (see [20]). The solution containing arbitrary functions of r with period i, the so-called the i-periodic constants, with the same potential was investigated in [21]. However, the using of such solution is suitable for the spectral problems only. Other forms of the QP equation with the Coulomb potential were considered in [22]. Applications of the factor (5) for describing some hadronic processes can be found in [23–25]. Recently, the relativistic S-factor (5) has been applied in [26] to reanalyze the mass limits obtained for magnetic monopoles which might have been produced at the Fermilab Tevatron.

We should like to remind that in the nonrelativistic approximation the total cross section in the leading order in the velocity expansion proportional to the resummation factor $S_{\rm nr} = |\psi(0)|^2$ [3]:

$$\frac{\sigma(e^+e^- \to f\bar{f})}{\sigma_{pt}} \propto \frac{v\,(3-v^2)}{2} S_{\rm nr}. \tag{6}$$

The resummation factors appears too in the parametrization of the imaginary part of the corresponding quark current correlators, in the Drell ratio $R(s)$ [10], which in the two-particle approximation can be approximated in terms of the Bethe–Salpeter (BS) amplitude of two charged particles $\chi_{\rm BS}(x)$ at $x = 0$ [27]. The nonrelativistic replacement of this amplitude by the wave function, which obeys the Schrödinger equation with the Coulomb potential (3), leads to formula (4) with a substitution $\alpha \to 4\alpha_{\rm S}/3$ for QCD. The possibility of using the RQP approach for our task is based on the fact that the BS amplitude, which parameterizes the physical quantity $R(s)$ and is taken at $x = 0$ and hence at the relative time $\tau = 0$, can be expressed in the case of the interaction of two relativistic particles of equal masses m through the wave RQP function in the momentum space, $\Psi_q(\mathbf{p})$, as

$$\chi_{\rm BS}(x=0) = \frac{1}{(2\pi)^3}\int d\Omega_{\mathbf{p}}\Psi_q(\mathbf{p}), \tag{7}$$

where $d\Omega_{\mathbf{p}} = (m d\mathbf{p})/E_p$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid $E_p^2 - \mathbf{p}^2 = m^2$.

The purpose of this paper is to generalize the previous study started in [16] to the case of the interaction of two particles of unequal masses ($m_1 \neq m_2$). The paper is organized as follows. In the next section, within the framework of the RQP approach in quantum field theory formulated in the relativistic configuration representation for the interaction of two relativistic particles of unequal masses [28] we derive the new relativistic S-factor. In Sec. III we analyze their behavior in the following cases: the nonrelativistic and relativistic cases, the case of equal masses, the case of one particle being at rest, and the ultrarelativistic case. Also, we compare the obtained factor with the factors considered in Refs. [3, 12–15]. Summarizing comments are given in Sec. IV.

2. RQP equation and the new relativistic S-factor in the case of two particles of unequal masses

The basis of our consideration is the completely covariant RQP equation into the momentum space constructed in [28] for the RQP wave function $\Psi_{q'}(\mathbf{p}')$ of two relativistic particles of unequal masses. This equation is given by

$$(2E_{q'} - 2E_{p'})\,\Psi_{q'}(\mathbf{p}') = \frac{2\mu}{m'(2\pi)^3}\int d\Omega_{\mathbf{k}'}\widetilde{V}\,(\mathbf{p}',\mathbf{k}';E_{q'})\,\Psi_{q'}(\mathbf{k}'), \tag{8}$$

where

$$d\Omega_{\mathbf{k}'} = \frac{m' d\mathbf{k}'}{E_{k'}}$$

is the relativistic three-dimensional volume element in the Lobachevsky space, $E_{k'} = \sqrt{m'^2 + \mathbf{k}'^2}$, $m' = \sqrt{m_1 m_2}$, and $\mu = m_1 m_2/(m_1+m_2)$ is the usual reduced mass.

Eq. (8) represents a relativistic generalization of the Lippmann-Schwinger equation in the spirit of the Lobachevsky geometry, which is realized on the upper half of the mass hyperboloid $E_{k'}^2 - \mathbf{k}'^2 = m'^2$. This equation describes the scattering over the quasipotential $\widetilde{V}\,(\mathbf{p}',\mathbf{k}';E_{q'})$

of an effective relativistic particle having mass m' and a relative 3-momentum $\mathbf{k}'$, emerging instead of the system of two particles and carrying the total c.m. energy of the interacting particles, $\sqrt{s}$, proportional to the energy $E_{k'}$ of one effective relativistic particle of mass m' (see [28, 29]):

$$\sqrt{s} = \sqrt{m_1{}^2 + \mathbf{k}^2} + \sqrt{m_2{}^2 + \mathbf{k}^2} = \frac{m'}{\mu} E_{k'}, \quad E_{k'} = \sqrt{m'^2 + \mathbf{k}'^2}. \tag{9}$$

The proper Lorentz transformations means a translation in the Lobachevsky space. The role of the plane waves corresponding to these translations are played by the following functions:

$$\xi(\mathbf{p}', \mathbf{r}) = \left(\frac{E_{p'} - \mathbf{p}' \cdot \mathbf{n}}{m'} \right)^{-1-irm'}, \tag{10}$$

where the module of the radius-vector, $\mathbf{r}$, ($\mathbf{r} = r\mathbf{n}, |\mathbf{n}| = \mathbf{1}$) is a relativistic invariant [29]. These functions correspond to the principal series of unitary representations of the Lorentz group and in the nonrelativistic limit ($p' \ll m$, $r \gg 1/m$) $\xi(\mathbf{p}', \mathbf{r}) \to \exp(i\mathbf{p}' \cdot \mathbf{r})$. The functions (10) satisfy the equation in terms of finite differences

$$\left(2E_{p'} - \hat{H}_0 \right) \xi(\mathbf{p}', \mathbf{r}) = 0. \tag{11}$$

Here

$$\hat{H}_0 = 2m' \left[\cosh \left(i\lambda' \frac{\partial}{\partial r} \right) + \frac{i\lambda'}{r} \sinh \left(i\lambda' \frac{\partial}{\partial r} \right) - \frac{\lambda'^2 \Delta_{\theta,\varphi}}{2r^2} \exp \left(i\lambda' \frac{\partial}{\partial r} \right) \right] \tag{12}$$

is the operator of the free Hamiltonian, while $\Delta_{\theta,\varphi}$ is its the angular part and $\lambda' = 1/m'$ is the Compton wavelengs associated with the effective relativistic particle of mass m'.

We note that Eq. (8) differs from of the QP equation considered in [30] by means of introduction into it of the relativistic reduced mass. However, in [30] was shown that it is possible to use the different expressions for the relativistic reduced mass by means of the choice of functional relationship between the relative 3-momentum $\mathbf{k}$ and the relativistic relative velocity of interacting particles, $\mathbf{v}$, connected, as is well-known, with their the total c.m. energy of interacting particles, $\sqrt{s}$, by relation (see, for instance, [14, 15])

$$|\mathbf{v}| = 2\sqrt{\frac{s - (m_1 + m_2)^2}{s - (m_1 - m_2)^2}} \left(1 + \frac{s - (m_1 + m_2)^2}{s - (m_1 - m_2)^2} \right)^{-1}. \tag{13}$$

In particular, if the dependence between the energy of the relative motion and the relativistic relative velocity $\mathbf{v}$ is given by expression (see [28, 29])

$$\frac{\mathbf{k}^{'2}}{2\mu} = \mu \left(\frac{1}{\sqrt{1 - |\mathbf{v}|^2}} - 1 \right), \tag{14}$$

this together with relation (13) leades to the expression (9). Such the choice of functionalrelationship has allowed to enter the concept of an effective relativistic particle, emerging instead of the system of two particles and having mass m', the relative 3-momentum $\mathbf{k}'$ and carrying the total c.m. energy of interacting particles, $\sqrt{s}$. Notice that the relative 3-momentum $\mathbf{k}^{'}$ of an effective relativistic particle, according to the expression (14), is invariant of the Loretz transformations.

The QP wave functions in the momentum space and relativistic configurational representation [28, 29] are related as follows:

$$\psi_{q'}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\Omega_{\mathbf{p}'} \xi(\mathbf{p}', \mathbf{r}) \Psi_{q'}(\mathbf{p}'), \quad \Psi_{q'}(\mathbf{p}') = \int d\mathbf{r} \xi^*(\mathbf{p}', \mathbf{r}) \psi_{q'}(\mathbf{r}). \tag{15}$$

For a spherically symmetric potential the application of transformations (15) (Shapiro transformations or ξ-transformations) to Eq. (8) leades to the equation, which is the integral form of the relativistic Schrödinger equation in the configurational representation:

$$\frac{1}{(2\pi)^3}\int d\Omega_{\mathbf{p}'}\,(2E_{q'}-2E_{p'})\,\xi(\mathbf{p}',\mathbf{r})\int d\mathbf{r}'\xi^*(\mathbf{p}',\mathbf{r}')\psi_{q'}(\mathbf{r}') = \frac{2\mu}{m'}V(r;E_{q'})\psi_{q'}(\mathbf{r}), \tag{16}$$

where the right-hand side is already local in the configuration representation and the transform of the potential, $V(r;E_{q'})$, is given in terms of the same relativistic plane waves.

We will introduce the new variables and functions:

$$\begin{gathered}
\mathbf{q}'=m'\mathbf{q},\mathbf{p}'=m'\mathbf{p},\mathbf{q}=\sinh(\chi_q)\mathbf{n}_q,\mathbf{p}=\sinh(\chi_p)\mathbf{n}_p,|\mathbf{n}_q|=|\mathbf{n}_p|=1,\\
\boldsymbol{\rho}=m'\mathbf{r},\boldsymbol{\rho}'=m'\mathbf{r}',\rho=|\boldsymbol{\rho}|,\rho'=|\boldsymbol{\rho}'|,r=|\mathbf{r}|,r'=|\mathbf{r}'|,d\mathbf{r}'=m'^{-3}d\boldsymbol{\rho}',\\
d\Omega_{\mathbf{p}'}=m'^3d\Omega_{\mathbf{p}},d\Omega_{\mathbf{p}}=\frac{d\mathbf{p}}{E_p},E_{q'}=m'E_q,E_{p'}=m'E_p,E_q=\sqrt{1+\mathbf{q}^2},E_p=\sqrt{1+\mathbf{p}^2},\\
V(r;E_{q'})=V(\rho/m';E_{q'}),\xi(\mathbf{p}',\mathbf{r})=(E_p-\mathbf{p}\cdot\mathbf{n})^{-1-i\rho}\equiv\xi(\mathbf{p},\boldsymbol{\rho}),\\
\psi_{q'}(\mathbf{r})=\psi_{m'q}(\boldsymbol{\rho}/m')\equiv\psi_q(\boldsymbol{\rho}),\Psi_{q'}(\mathbf{p}')\equiv m'^{-3}\Psi_q(\mathbf{p}).
\end{gathered}\tag{17}$$

By using the expansions

$$\begin{gathered}
\xi(\mathbf{p},\boldsymbol{\rho})=\sum_{\ell=0}^{\infty}(2\ell+1)i^\ell p_\ell(\rho,\cosh\chi_p)P_\ell\left(\frac{\mathbf{p}\cdot\boldsymbol{\rho}}{p\rho}\right),\\
\psi_q(\boldsymbol{\rho})=\sum_{\ell=0}^{\infty}(2\ell+1)i^\ell\frac{\varphi_\ell(\rho,\chi_q)}{\rho}P_\ell\left(\frac{\mathbf{q}\cdot\boldsymbol{\rho}}{q\rho}\right),
\end{gathered}\tag{18}$$

and also formula [19]

$$p_\ell(\rho,\cosh\chi)=\frac{(-1)^\ell(\sinh\chi)^\ell}{\rho^{(\ell+1)}}\left(\frac{d}{d\cosh\chi}\right)^\ell\left(\frac{\sin\rho\chi}{\sinh\chi}\right),$$

the integral Eq. (16) transformed to the form

$$\begin{gathered}
\frac{2}{\pi}\int_0^\infty d\chi'\frac{(\sinh\chi')^{2\ell+2}(-1)^{\ell+1}}{\rho^{(\ell+1)}}\,(2\cosh\chi-2\cosh\chi')\left(\frac{d}{d\cosh\chi'}\right)^\ell\left(\frac{\sin\rho\chi'}{\sinh\chi'}\right)\times\\
\times\left(\frac{d}{d\cosh\chi'}\right)^\ell\frac{1}{\sinh\chi'}\int_0^\infty d\rho'\frac{\rho'\sin\rho'\chi'}{(-\rho')^{(\ell+1)}}\varphi_\ell(\rho',\chi)=\frac{2\mu}{m'^2}\frac{V(\rho/m';E_q)\varphi_\ell(\rho,\chi)}{\rho}.
\end{gathered}\tag{19}$$

Here $P_\mu^\nu(z)$ is a Legendre function of the first kind, and the function

$$p_\ell(\rho,\cosh\chi)=\frac{(-1)^{\ell+1}}{\rho}\sqrt{\frac{\pi}{2\sinh\chi}}(-\rho)^{(\ell+1)}P_{-1/2+i\rho}^{-1/2-\ell}(\cosh\chi) \tag{20}$$

is the solution of Eq. (11); χ' and χ are the rapidities which are related to E_p and E_q as $E_p=\cosh\chi', E_q=\cosh\chi$, and the function

$$(-\rho)^{(\ell+1)}=i^{\ell+1}\frac{\Gamma(\ell+1+i\rho)}{\Gamma(i\rho)} \tag{21}$$

is the generalized power [19] where $\Gamma(z)$ is the gamma-function.

For the first time, the solution of Eq. (19) in the case of the interaction of two relativistic particles of equal masses at $\ell = 0$, not containing the i-periodic constants, was received in [16]. This approach leades to the relativistic S-factor (5).

We note that the applying of ξ-transformation (15) to the Coulomb interaction (3) gives the potential in momentum space (see [16])

$$V(\mathbf{\Delta}) \sim \frac{1}{\chi_{\mathbf{\Delta}} \sinh \chi_{\mathbf{\Delta}}},$$

where the relative rapidity $\chi_{\mathbf{\Delta}}$ corresponds to $\mathbf{\Delta} = \mathbf{p}'(-)\mathbf{k}'$ and is defined in terms of the square of the momentum transfer by $Q^2 = -(p' - k')^2 = 2(\cosh \chi_{\mathbf{\Delta}} - 1)$. For large Q^2 the potential $V(\mathbf{\Delta})$ behaves as $(Q^2 \ln Q^2)^{-1}$, which reproduces the principal behaviour of the QCD potential proportional to $\bar{\alpha}_S(Q^2)/Q^2$ with $\bar{\alpha}_S(Q^2)$ being the QCD running coupling. This property of the potential (3), its QCD-like behaviour, was noted in [20].

Generalising the method developed in [16] (see also [31, 32]), we will seek a solution of RQP equation (19) with the potential (3) in the form

$$\varphi_\ell(\rho, \chi) = \frac{(-\rho)^{(\ell+1)}}{\rho} \int\limits_{\alpha_-}^{\alpha_+} d\zeta e^{i\rho\zeta} R_\ell(\zeta, \chi), \tag{22}$$

where the ζ-integration is performed in the complex plane over a contour with end points α_- and α_+ (see Fig. 1).

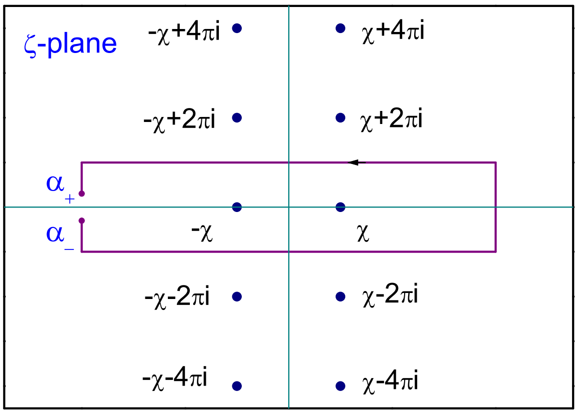

FIG. 1. *Contour of integration in Eq. (22) and singularities of the function (25) in the complex ζ-plane.*

Substituting (22) into (19) at $\ell = 0$ and taking into account that

$$\frac{1}{i\pi} \int\limits_0^\infty d\rho' \sin(\rho'\chi') e^{i\rho'\zeta} = \frac{1}{i\pi} \frac{\chi'}{\chi'^2 - \zeta^2},$$

we arrive to the following equation

$$\frac{d}{d\zeta}\Big[(\cosh \chi - \cosh \zeta)\, R_0(\zeta, \chi)\Big] - \frac{i\alpha\mu}{m'} R_0(\zeta, \chi) = 0, \tag{23}$$

with the boundary condition

$$e^{i\rho\zeta} (\cosh \chi - \cosh \zeta)\, R_0(\zeta, \chi)\Big|_{\zeta=\alpha_-}^{\zeta=\alpha_+} = 0. \tag{24}$$

As a result the solution of Eq. (23) is

$$R_0(\zeta,\chi) = C_0(\chi)\frac{e^{\zeta}}{(e^{\zeta}-e^{\chi})^2}\left[\frac{e^{\zeta}-e^{-\chi}}{e^{\zeta}-e^{\chi}}\right]^{-1+iA}, \tag{25}$$

where $A = \alpha\mu/(m'\sinh\chi)$, and $C_0(\chi)$ is an arbitrary function of χ.

The branch points of the function (25) are $\pm\chi + 2\pi ni$ (see Fig. 1). The contour of integration must not intersect cuts which we take from $-\infty + 2\pi ni$ to $\pm\chi + 2\pi ni$. In the case when the interaction vanishes, $\alpha \to 0$, the solution $\varphi_\ell(\rho,\chi)$ should reproduce the known free wave function:

$$\lim_{\alpha\to 0}\varphi_\ell(\rho,\chi) = \rho p_\ell(\rho,\cosh\chi) \xrightarrow[\rho\to\infty]{} \frac{\sin(\rho\chi - \pi\ell/2)}{\sinh\chi}, \tag{26}$$

where the function $p_\ell(\rho,\cosh\chi)$ is determined in (20). Taking into account these remarks and the boundary condition (24), we take $\alpha_- = -R - i\varepsilon, \alpha_+ = -R + i\varepsilon$ with $R \to +\infty, \varepsilon \to +0$. The vertical part of the contour to the right is given by $\mathrm{Re}\zeta = +R$. It is also convenient for finding a connection to an integral representation of the hypergeometric function to take the horizontal parts of the contour to be characterized by $\mathrm{Im}\zeta = \pm\pi$ (see Fig. 1).

Substituting the solution (25) into (22) at $\ell = 0$ and performing ζ-integration in the complex plane along a contour with end points α_- and α_+ (in the same way as in [16, 31, 32]) we obtain the resulting solution which does not contain the i-periodic constant in terms of hypergeometrical function as

$$\varphi_0(\rho,\chi) = -N_0(\chi)(-\rho)^{(1)}e^{i\rho\chi + iA\chi}F\left(1-iA, 1-i\rho; 2; 1-e^{-2\chi}\right). \tag{27}$$

The normalization constant $N_0(\chi)$ in Eq. (27) can be obtained (also as in [16]) from the condition (26) at $\ell = 0$.

We should like to remind that the Bethe-Salpeter amplitude $\chi_{\mathrm{BS}}(x=0)$ is associated with the RQP wave function in the momentum space, $\Psi_q(\mathbf{p})$, by the relation (7). Taking into account the transformations (15) and notations (17), the relationship of the Bethe-Salpeter amplitude with the RQP wave function, $\psi_q(\boldsymbol{\rho})$, is

$$\chi_{\mathrm{BS}}(x=0) = \psi_q(\boldsymbol{\rho})|_{\rho=i}.$$

The generalized power (21) in the solution (22) vanishes at $\rho = i$ for all $\ell \neq 0$. Thus, the expansion (18) for the wave function $\psi_q(\boldsymbol{\rho})$ contains only s-wave ($\ell = 0$). Hence, by using relations (26) and (27) we can calculate $|\psi_q(i)|^2$, which leads to the following expression for the relativistic S-factor in the case of two particles of unequal masses:

$$S_{\mathrm{RQP}}(\chi) = \lim_{\rho\to i}\left|\frac{\varphi_0(\rho,\chi)}{\rho}\right|^2 = \frac{X_{\mathrm{RQP}}(\chi)}{1-\exp\left[-X_{\mathrm{RQP}}(\chi)\right]}, \quad X_{\mathrm{RQP}}(\chi) = \frac{2\pi\alpha\mu}{m'\sinh\chi}, \tag{28}$$

where χ is the rapidity which is related to the total c.m. energy, $\sqrt{s}$, as $(m'^2/\mu)\cosh\chi = \sqrt{s}$.

The functions $\sinh\chi$ and $X_{\mathrm{RQP}}(\chi)$ in Eq. (28) can be expressed in terms of the "velocity" u determined by the relation

$$u = \sqrt{1 - \frac{4m'^2}{s-(m_1-m_2)^2}}, \tag{29}$$

in the form

$$\sinh\chi = \frac{2\mu u}{m'\sqrt{1-u^2}}, \quad X_{\mathrm{RQP}}(\chi) = \frac{\pi\alpha\sqrt{1-u^2}}{u}. \tag{30}$$

We note that the square of relative 3-momentum $\mathbf{k}'$ for an effective relativistic particle, having mass m', the total c.m. energy of interacting particles, $\sqrt{s}$, and emerging instead of the system of two particles, is defined by formula (9) and connected with the relative relativistic velocity of interacting particles, $\mathbf{v}$, by the expression (14). In turn the relative velocity $\mathbf{v}$ of interacting particles is expressed through their the total c.m. energy of interacting particles, $\sqrt{s}$, by relation (13). Thence, taking into consideration the determination (29), we find

$$|\mathbf{v}| = \frac{2u}{1+u^2}. \tag{31}$$

Then expressions (14) and (31) give

$$\mathbf{k}'^2 = \mu^2 (u'_{\rm rel})^2, \tag{32}$$

where μ is the usual reduced mass, and

$$u'_{\rm rel} = \frac{2u}{\sqrt{1-u^2}} \tag{33}$$

is the relative velocity of an effective relativistic particle with mass m' emerging instead of the system of two particles. This result is found to be in full agreement with the physical meaning of Eq. (8), which is a relativistic generalization of the Lippmann-Schwinger equation in the spirit of Lobachevsky geometry. Notice that the 3-momentum $\mathbf{k}'$ of an effective relativistic particle and hence its relative velocity (33), according to Eqs. (14) and (32), are an invariants of the Lorentz transformations.

Thus, in terms of relative velocity of an effective relativistic particle (33), the S-factor (28) gives by expression

$$S_{\rm RQP}(u'_{\rm rel}) = \frac{X_{\rm RQP}(u'_{\rm rel})}{1-\exp\left[-X_{\rm RQP}(u'_{\rm rel})\right]}, \quad X_{\rm RQP}(u'_{\rm rel}) = \frac{2\pi\alpha}{u'_{\rm rel}}. \tag{34}$$

The S-factor in Eq. (34) only formally has the same form, as the nonrelativistic S-factor (4). However, the S-factor in Eq. (34) has an obviously relativistic nature since as the argument r (the module of radius-vector $\mathbf{r}$) in the Coulomb potential (3) and the relativistic relative velocity of interacting particles, $\mathbf{v}$, (see [29]) both are relativistic invariants and hence the relative velocity of an effective relativistic particle (33), according to Eqs. (14) and (32), possesses this property as well.

3. Analysis relativistic threshold resummation S-factor

The relativistic threshold resummation factor (28) [or (34)] has the following important properties:

• In the nonrelativistic limit, $u \ll 1$, it reproduces the well-known nonrelativistic result.

• In the relativistic limit, $u \to 1$, the factor (28) [or (34)] goes to unity.

• In the case of equal masses it coincides with S-factor (5).

• The case when one of the particles is at rest means that $m_1 \to \infty$. Todorov [33] suggested that any QP equation for scalar particles in the case m_1 (or m_2) $\to \infty$ has to give the same result as the Klein–Gordon equation with the Coulomb interaction. This means that the relative relativistic velocity of interacting particles, $\mathbf{v}$, in this case has to be equal to the relativistic velocity of the "light" particle:

$$|\mathbf{v}| \xrightarrow[m_1\to\infty]{} \frac{|\mathbf{k}|}{\sqrt{{m_2}^2+\mathbf{k}^2}}. \tag{35}$$

For the "velocity" u this gives the following limited expression

$$u \xrightarrow[m_1\to\infty]{} \frac{|\mathbf{k}|}{\sqrt{{m_2}^2+\mathbf{k}^2}+m_2},$$

that according the expression in (31) leades to (35).

• In the ultrarelativistic limit, as it was argued in Refs. [34, 35], the bound state spectrum vanishes since mass of an effective relativistic particle $m' \to 0$. This feature reflects an essential difference between potential models and quantum field theory where an additional dimensional parameter Λ appears. One can conclude that within a potential model, the S-factor which correspond to the continuous spectrum should go to unity in the limit $m' \to 0$. Thus, in contrast to the nonrelativistic case, the relativistic resummation factor, the S-factor (28) [or (34)] reproduces the known nonrelativistic and the expected ultrarelativistic limits.

To illustrate the differences between the nonrelativistic S-factor (4) and the new relativistic S-factor in Eq. (34) in more detail, in Fig. 2 we plot the behavior of these factors as functions of u at different values of the parameter α (the numbers at the curves). The solid lines correspond

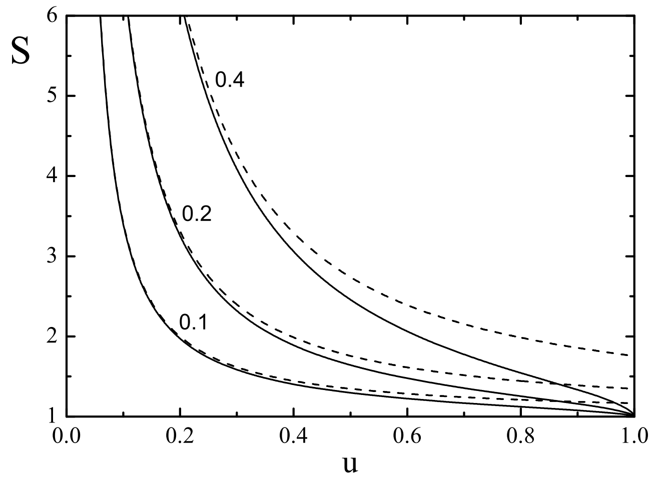

FIG. 2. *Behavior of the S-factor at different values of the parameter α (the numbers at the curves). The solid lines correspond to the new relativistic S-factor (34) and the dashed lines to the nonrelativistic S-factor (4)*

to the S-factor in Eq. (34) and the dashed lines to the S-factor (4) with a substitution $v_{\rm nr} \to u$. From this figure one can see that in the region of nonrelativistic values of u, $u \le 0.2$, where the influence of the S-factor is big, the difference between (34) and (4) is practically absent. However, when α increases, the nonrelativistic expression (4) gives a less suitable result in the region of large values u, in particular, as $u \to 1$. Thus, the above analysis demonstrates that the relativistic S-factor in Eq. (34), as would be expected, coincides in form with the nonrelativistic S-factor (4). However, the relative velocity of an effective relativistic particle (33) emerging instead of the system of two particles, now plays role of the parameter of velocity, but not the relativistic relative velocity of interacting particles, $\mathbf{v}$.

It should be stressed that the S-factor in Eq. (34) differs from the S-factor for the interaction of two relativistic particles of unequal masses

$$S_{\rm A}(|\mathbf{v}|) = \frac{X_{\rm A}(|\mathbf{v}|)}{1 - \exp\left[-X_{\rm A}(|\mathbf{v}|)\right]}, \qquad X_{\rm A}(|\mathbf{v}|) = \frac{2\pi\alpha}{|\mathbf{v}|}, \tag{36}$$

which was obtained in Ref. [15], by the meaning of the relativistic relative velocity, $\mathbf{v}$, which here is given by expression (13). This the S-factor was derived within the framework of relativistic quantum mechanics on the basis of the Schrödinger equation in suggestion that the 3-momentum of the relativistic particles, $\mathbf{p_1}, \mathbf{p_2}$, will satisfy condition $\mathbf{p_2} = -\mathbf{p_1} m_2/m_1 \Leftrightarrow \mathbf{v_1} = -\mathbf{v_2}$.

In the case of equal masses ($m_1 = m_2 = m$), the new S-factor (34) differs also from the factor

$$S_{\rm H}(v_{\rm rel}) = \frac{X_{\rm H}(v_{\rm rel})}{1 - \exp\left[-X_{\rm H}(v_{\rm rel})\right]}, \quad X_{\rm H}(v_{\rm rel}) = \frac{2\pi\alpha}{v_{\rm rel}}, \quad v_{\rm rel} = \frac{2v}{1+v^2}, \tag{37}$$

which was presented in Ref. [3]. The same form of the S-factor can be found in earlier paper [12]. Indeed, the factors (36) and (37) in form and in the nonrelativistic limit ($|\mathbf{v}|, v_{\rm rel}, u \to 0$) coincide with the factor (34). However, the relativistic limits ($|\mathbf{v}|, v_{\rm rel} \to 1$) of the factors (36) and (37) differ essentially from the relativistic limit of the factor (34) equal to unity as $u \to 1$. Furthermore, in the case of the interaction of two relativistic particles of equal masses the S-factor (34) differs from the factor

$$K = G(\eta)\kappa, \tag{38}$$

where

$$G(\eta) = \frac{2\pi\eta}{1-\exp(-2\pi\eta)}, \tag{39}$$

which was obtained in Ref. [14], not only in form but also in a different behavior in the nonrelativistic ($v_{\rm rel}, u \to 0$) and the relativistic ($v_{\rm rel}, u \to 1$) limits (see Figs. 3 and 4). The function

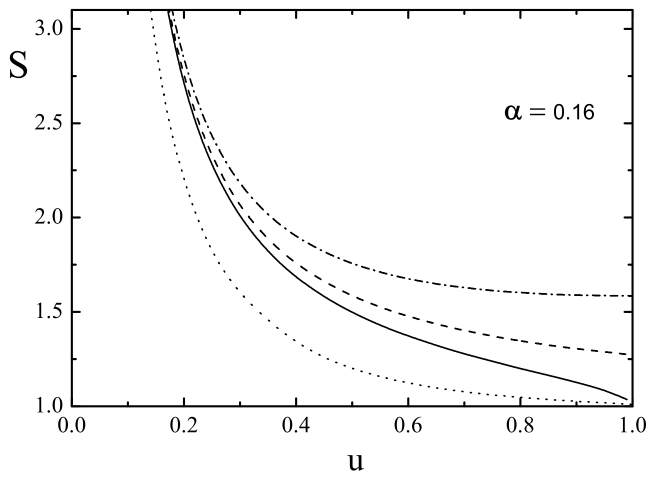

FIG. 3. *Comparison of the S-factor behavior. The solid curve corresponds to the relativistic S-factor (34). The nonrelativistic S-factor (4) is presented by the dashed line, the factor (36) is presented by the dash-dotted line, and the factor (38) is shown as the dotted line.*

$G(\eta)$ in Eq. (39) is the nonrelativistic S-factor (4) with $\eta = \alpha/v_{\rm rel}$, and κ is a correction factor the expression for which contains the series and infinite products as multipliers and is highly cumbersome. According to Ref. [14], the influence of the κ is essential since $\kappa \to 1$ only as $\alpha \to 0$.

To illustrate the difference between the above factors, we show in Fig. 3 the behavior of the S-factors (4), (34), (36) and (38) as functions of u for a fixed value of $\alpha = 0.16$. The solid line represents the relativistic S-factor (34); the dashed line the nonrelativistic S-factor (4); the dotted line the factor defined by formula (38); the dash-dotted the factor (36). This figure demonstrates that the S-factors considered have an essentially different behavior as $u \to 1$.

To compare the relativistic factors (34) and (38) in more detail, we plot in Fig. 4 the ratio, which is denoted as $N(\eta)$, of the relativistic factor (34) or (38) to the nonrelativistic S-factor (39) for different values of α (the numbers at the curves). The solid lines correspond to the S-factor (34). The dashed lines, which are taken from Ref. [14], represent the ratio of the factor (38) to the nonrelativistic S-factor. As can be seen from Fig. 4, there is an essential difference between the relativistic factors (34) and (38). For example, in the nonrelativistic limit, when η increases ($v_{\rm rel} \to 0$), the relativistic S-factor (38) reproduces the nonrelativistic limit only as $\alpha \to 0$ (see Ref. [14] for additional details).

Thus, the above analysis demonstrates that the relativistic S-factor (34), as would be expected, coincides in form with the nonrelativistic S-factor (4). However, the relative velocity of an effective relativistic particle (33) emerging instead of the system of two particles, now plays role of the parameter of velocity, but not the relativistic relative velocity of interacting particles, $|\mathbf{v}|$.

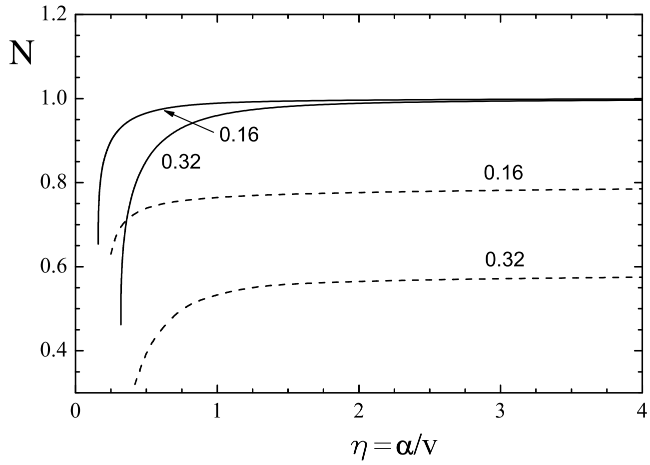

FIG. 4. *Ratio of relativistic S-factor to nonrelativistic one, $N(\eta)$, for different values of α (the numbers at the curves). The solid lines correspond to the S-factor (34) and the dashed lines taken from Ref. [14] correspond to the factor (38).*

4. Conclusion

The new relativistic threshold resummation S-factor (34) for the interaction of two relativistic particles of unequal masses was obtained. For this aim the RQP equation in relativistic configuration representation [28] with the Coulomb potential for the interaction of two relativistic particles of unequal masses was used.

The new relativistic threshold resummation factor obtained here reproduce both the known nonrelativistic and expected ultrarelativistic limits and correspond to the QCD-like Coulomb potential. The Coulomb potential only formally has the same form as the nonrelativistic potential but differs in the relativistic configuration representation since its behavior corresponds to the quark-antiquark potential $V_{q\bar{q}} \sim \bar{\alpha}_S(Q^2)/Q^2$ with the invariant charge $\bar{\alpha}_S(Q^2) \sim 1/\ln Q^2$. So, the principal effect coming from the running of the QCD coupling is accumulated.

The new S-factor coincides in form with the nonrelativistic S-factor (4); however, the role of the parameter of velocity is played not by the relative relativistic velocity of interacting particles, $\mathbf{v}$, but by the relative velocity (33) of an effective relativistic particle emerging instead of the system of two particles.

Our analysis has shown that the new expression (34) differs significatly (see Figs. 2–4) from expressions (4) and (36)–(38) which were proposed earlier. As the new relativistic resummation factor (34) was obtained within the framework of completely covariant method, one can expect that this factor takes better into account relativistic character of interacting particles.

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