superposition of a non-local separable interaction and a local quasipotential not admitting bound states

Yu.D. Chernichenko

Gomel State Technical University, 246746 Gomel, Belarus.
E-mail: chern@gstu.gomel.by

Abstract

Within the relativistic quasipotential approach to quantum field theory, a method of solving a finite-difference quasipotential equation involving a non-local separable quasipotential simulating the interaction between two relativistic spinless particles of unequal masses is generalized to the case where the total interaction is the superposition of a non-local separable quasipotential and a local one. Besides, the local component of the total interaction is supposed to be known and that it can not admit bound states. This has permitted to find an explicit expression for the additional phase-shift, to determine the conditions under which bound states may exist, and to generalize the Levinson theorem.

1 Introduction

Non-local separable potentials are widely used in nuclear physics and in many-body problems. In particular, non-local separable interactions were used in solving Faddeev equations to the three-body problem. The given approach proved to be fruitful in solving inverse problems [1–4]. However, this approach cannot be applied to essentially relativistic systems [5, 6]. For instance, for systems consisting of light quarks, the contribution of relativistic corrections to the interaction Hamiltonian is comparable to the main, non-relativistic, term. A relativistic description is also necessary for investigation of radiative decays of mesons and nucleon resonances, where the energy of the emitted photon may be comparable with or even larger than the constituent quark mass.

The quasipotential approach proposed in [7] proved to be an effective tool for constructing a relativistic description of two-particle systems [8–11]. In the present study, a method for solving a finite-difference quasipotential equation in configuration space is constructed within the relativistic quasipotential approach to quantum field theory developed in [12]. It is designed for the case when the total quasipotential simulating the interaction between two relativistic spinless particles of unequal masses $m_1 \neq m_2$ is the superposition of a non-local separable quasipotential and a local one.

The necessity of such interaction is a consequence of the meson theory of nuclear forces. This theory suggests that the interaction between two nucleons is local at large distances, but becomes non-local and singular if two nucleons come close together. Besides, we can suppose that the local part $w(\rho)$ of the total quasipotential is determined to agree with experimental data at low energies. Because of the absence of exact theoretical information about the nuclear forces at small distances, it is likely to assume that, near the origin, the non-local component of total quasipotential is separable.

We take a total quasipotential of the form:

$$V(\vec{\rho},\vec{\rho}';E_{q'}) \equiv V(\vec{\rho},\vec{\rho}') = w(\rho)\delta(\vec{\rho}'-\vec{\rho}) + \sum_{l=0}^{\infty}(2l+1)\varepsilon_l v_l(\rho)v_l(\rho')P_l\left(\frac{\vec{\rho}\cdot\vec{\rho}'}{\rho\rho'}\right), \qquad (1)$$

$\varepsilon_l = \pm 1, \rho = |\vec{\rho}|, \rho' = |\vec{\rho}'|$.

Here, $\varepsilon_l = 1$ corresponds to a repulsive quasipotential and $\varepsilon_l = -1$ to attractive one; $P_l(z)$ is a Legendre function of the first kind. Accordingly, in the system of units where $\hbar = c = 1$, the relativistic analogy of the differential Schrödinger equation for the wave function $\Psi_{q'}(\vec{\rho})$ in configuration space with a quasipotential (1) constructed in [13] is given by

$$\frac{m'^2}{\mu}\left[\cosh\left(i\lambda'\frac{\partial}{\partial\rho}\right)+\frac{i\lambda'}{\rho}\sinh\left(i\lambda'\frac{\partial}{\partial\rho}\right)-\frac{\lambda'^2}{2\rho^2}\Delta_{\theta,\varphi}\exp\left(i\lambda'\frac{\partial}{\partial\rho}\right)-\cosh\chi'\right]\Psi_{q'}(\vec{\rho})+ \tag{2}$$
$$+\int d\vec{\rho}' V(\vec{\rho},\vec{\rho}')\Psi_{\rho'}(\vec{\rho}')=0,$$

where $\Delta_{\theta,\varphi}$ is the angular part of the Laplace operator, $\lambda'=1/m'$ is the Compton wavelength connected with the effective relativistic particle of mass $m'=\sqrt{m_1m_2}$, and $\mu=m'^2/(m_1+m_2)$.

Obviously, Eq.(2) describes scattering of an effective relativistic particle of mass m' with a relative 3-momentum $\vec{q}'$ on the quasipotential (1) and the total c.m. energy of the particles which is proportional to the energy of one effective relativistic particle of mass m' [13] ,

$$\sqrt{S}_{q'}=(m'/\mu)E_{q'}=(m'^2/\mu)\cosh\chi',\quad E_{q'}=\sqrt{m'^2+\vec{q}'^2}. \tag{3}$$

Expanding the wave function $\Psi_{q'}(\vec{\rho})$ in terms of partial waves [14] as

$$\Psi_{q'}(\vec{\rho})=\sum_{l=0}^{\infty}(2l+1)i^l\frac{\psi_l(\rho,\chi')}{\rho}P_l\left(\frac{\vec{q}'\cdot\vec{\rho}}{q'\rho}\right),\quad q'=|\vec{q}'|, \tag{4}$$

we can recast Eq.(2) into the form

$$\left[\nabla+\left(1+\frac{l(l+1)}{r^{(2)}}\right)\nabla^*-2\cosh\chi'+W(r)\right]\psi_l(r,\chi')+\frac{1}{2}\varepsilon_l V_l(r)\int_0^{\infty}dr'V_l(r')\psi_l(r',\chi')=0, \tag{5}$$

where $\nabla=\exp\left(-i\frac{d}{dr}\right)$, $\nabla^*=\exp\left(i\frac{d}{dr}\right)$, $V_l(r)=\sqrt{8\pi\lambda'\mu/m'^2}\rho v_l(\rho)$,

$$W(r)=2\mu w(\rho)/m'^2,\quad r^{(2)}=r(r+i),\quad \rho=\lambda' r,\quad \rho'=\lambda' r'.$$

Thus, the fact that, within the relativistic quasipotential approach, the total c.m. energy of two relativistic spinless particles of unequal masses can be represented in a form proportional to the energy of one effective relativistic particle of mass m', makes it possible to reduce the relativistic problem of two bodies of unequal masses to an one-body problem. The present study is devoted to solving Eq.(5) with the boundary condition

$$\psi_l(0,\chi')=0, \tag{6}$$

to obtaining the expression for the additional phase-shift, to investigating the conditions of existence bound states and to generalizing the Levinson theorem in the case of the superposition of a non-local separable quasipotential and a local one. Besides, the local part $W(r)$ of the total quasipotential, being known from the low-energy experimental data, does not admit bound states.

2 Some properties of regular solution and the spectral density for the local quasipotential

In order that Eq.(5) with the boundary condition (6) have a unique solution, $V_l(r)$ and $W(r)$ must satisfy conditions

$$rV_l(r)\in L_1(0,\infty),\quad rW(r)\in L_1(0,\infty). \tag{7}$$

We further introduce the regular solution $\varphi_l(r,\chi')$ satisfying the boundary condition

$$\varphi_l(0,\chi') = 0 \tag{8}$$

and the Jost solution $f_l(r,\chi')$ of Eq.(5) at $\varepsilon_l \equiv 0$:

$$\lim_{r\to 0} Q_l(\coth\chi')\varphi_l(r,\chi')/s_l(r,\chi') = 1, \tag{9}$$

$$\lim_{r\chi'\to\infty} f_l(r,\chi')/e_l^{(1)}(r,\chi') = 1, \tag{10}$$

where

$$s_l(r,\chi') \underset{r\to 0}{\approx} e^{i\pi(l+1)}Q_l(\coth\chi')(-r)^{(l+1)}/\Gamma(l+1), \tag{11}$$

$$e_l^{(1)}(r,\chi') \underset{r\chi'\to\infty}{\approx} \exp[i(r\chi' - \pi l/2)]. \tag{12}$$

Here, $\Gamma(z)$ is the gamma function, $Q_l(z)$ is the Legendre function of the second kind, and

$$(-r)^{(\lambda)} = i^\lambda \Gamma(ir+\lambda)/\Gamma(ir).$$

It should be noted that the functions $s_l(r,\chi')$ and $e_l^{(1)}(r,\chi')$ [15, 16] are solutions of Eq.(5) in the case when the interaction is switched off ($W(r) \equiv 0, \varepsilon_l \equiv 0$).

The regular solution $\varphi_l(r,\chi')$ can be represented in the form

$$\varphi_l(r,\chi') = \frac{1}{2iQ_l(\coth\chi')}\left[F_l^W(-\chi')f_l(r,\chi') + e^{i\pi(l+1)}F_l^W(\chi')f_l(r,-\chi')\right], \tag{13}$$

where $F_l^W(\chi')$ is the Jost function of the local quasipotential $W(r)$ connected with its phase-shift $\delta_l^W(\chi')$ and the Jost solution by the expressions

$$F_l^W(\chi') = \left|F_l^W(\chi')\right|\exp[-i\delta_l^W(\chi')], \tag{14}$$

$$F_l^W(\chi') = \lim_{r\to 0}\frac{e^{-i\pi l}(2l+1)Q_l(\coth\chi')}{\Gamma(l+1)\sinh\chi'(-r)^{(-l)}} f_l(r,\chi').$$

Besides, the Jost function has the following property:

$$\left[F_l^W(\chi')\right]^* = F_l^W(-\chi'^*). \tag{15}$$

Hence it follows that

$$\left[\delta_l^W(\chi')\right]^* = \delta_l^W(\chi'^*), \quad \delta_l^W(-\chi') = -\delta_l^W(\chi'). \tag{16}$$

It is easily seen from (10), (12)–(16) that

$$\varphi_l(r,\chi') \underset{r\chi'\to\infty}{\approx} \frac{\left|F_l^W(\chi')\right|}{Q_l(\coth\chi')}\sin\left[r\chi' - \frac{\pi l}{2} + \delta_l^W(\chi')\right]. \tag{17}$$

Finally, in absence of bound states of $W(r)$ the regular solution $\varphi_l(r,\chi')$ satisfies the orthogonality property

$$\int_0^\infty dr\varphi_l(r,\chi)\varphi_l^*(r,\chi') = \frac{\delta(\cosh\chi' - \cosh\chi)}{d\rho_l(\cosh\chi')/d(\cosh\chi')}, \tag{18}$$

where

$$\frac{d\rho_l(\cosh\chi')}{d(\cosh\chi')} = \frac{2}{\pi\sinh\chi'}\left|Q_l(\coth\chi')/F_l^W(\chi')\right|^2, \quad E' = E_{q'}/m'_* = \cosh\chi' \geq 1, \tag{19}$$

is the spectral density. On the other hand, the spectral function $\rho_l(\cosh\chi')$ enables us to get the completeness property

$$\int_1^\infty d\rho_l(\cosh\chi')\varphi_l(r,\chi')\varphi_l^*(r',\chi') = \delta(r'-r). \tag{20}$$

The relations (18) and (20) are obvious generalizations of the orthogonality and completeness properties for the functions $s_l(r,\chi')$ [16], and reduces to them when $W(r)\equiv 0$.

3 Wave function and phase-shift for the superposition of a non-local separable and a local quasipotentials

The properties (18) and (20) permit us to introduce the relativistic integral transformations

$$\tilde{\psi}_l(\chi',\chi) = \int_0^\infty dr\psi_l(r,\chi')\varphi_l^*(r,\chi), \tag{21}$$

$$\psi_l(r,\chi') = \int_1^\infty d\rho_l(\cosh\chi)\tilde{\psi}_l(\chi',\chi)\varphi_l(r,\chi), \tag{22}$$

$$\tilde{V}_l(\chi) = \int_0^\infty drV_l(r)\varphi_l^*(r,\chi), \tag{23}$$

$$V_l(r) = \int_1^\infty d\rho_l(\cosh\chi)\tilde{V}_l(\chi)\varphi_l(r,\chi). \tag{24}$$

It is necessary to note that the integral transformations (21) – (24) are generalizations of the relativistic integral Hankel transformations introduced in [14], and reduce to them when $W(r)\equiv 0$.

The total phase-shift $\delta_l(\chi')$ is represented in the form

$$\delta_l(\chi') = \delta_l^W(\chi') + \delta_l^V(\chi'), \tag{25}$$

where $\delta_l^V(\chi')$ is the additional phase-shift due to the separable quasipotential, which also depends on $W(r)$, and should not be confused with the phase-shift of the separable quasipotential. Using transformations (22) and (24), the equation (5) reduces to

$$(\cosh\chi' - \cosh\chi)\tilde{\psi}_l(\chi',\chi) = \frac{1}{2}\varepsilon_l N_l(\chi')\tilde{V}_l(\chi), \tag{26}$$

where

$$N_l(\chi') = \int_0^\infty dr' V_l(r')\psi_l(r',\chi'). \tag{27}$$

Let us now set

$$V_l(r) = \nu_l(r)U_l(r), \tag{28}$$

where

$$\nu_l(r) = \exp[i\pi(l+1)](r)^{(l+1)}/(-r)^{(l+1)}.$$

Suppose that the relativistic integral transformation is valid for the function $U_l(r)$:

$$\tilde{U}_l(\chi) = \int_0^\infty drU_l(r)\varphi_l^*(r,\chi), \tag{29}$$

$$U_l(r) = \int_1^\infty d\rho_l(\cosh\chi)\tilde{U}_l(\chi)\varphi_l(r,\chi). \quad (30)$$

We further suppose that for real-valued l and χ' we may write:

$$[\varphi_l(r,\chi')]^* = \nu_l(r)\varphi_l(r,\chi').$$

Instead of (27), we therefore have

$$N_l(\chi') = \int_1^\infty d\rho_l(\cosh\chi)\tilde{\psi}_l(\chi',\chi)\tilde{U}_l(\chi). \quad (31)$$

Now note that the conditions in (7) means that the function $\tilde{V}_l(\chi)$ is continuous everywhere, whereas the function $Q_l(\coth\chi)\tilde{V}_l(\chi)/|F_l^W(\chi)|$ is differentiable for all nonnegative χ. Moreover, it follows from (23) that

$$Q_l(\coth\chi)\tilde{V}_l(\chi)/|F_l^W(\chi)| = O(1), \quad |\chi| \to \infty; \quad (32)$$

$$\tilde{V}_l(\chi) = O(1), \quad \chi \to 0, \quad (33)$$

provided that the condition (7) is satisfied. It is obvious that, by virtue of (28), the function $\tilde{U}_l(\chi)$ also possesses the aforementioned property.

For scattering states ($E' = E_{q'}/m' = \cosh\chi' \geq 1$), solution of equation (26) is given by

$$\tilde{\psi}_l(\chi',\chi) = \frac{\delta(\cosh\chi' - \cosh\chi)}{d\rho_l(\cosh\chi')/d(\cosh\chi')} + \frac{1}{2}\varepsilon_l N_l(\chi')\mathrm{P}\frac{\tilde{V}_l(\chi)}{\cosh\chi' - \cosh\chi}, \quad (34)$$

where P means the principla value. The factor in front of the δ-function was chosen in accordance with the normalization of the wave function; that is, in the absence of separable interaction ($\varepsilon_l \equiv 0$), the representation (22) must lead to the regular solution $\varphi_l(r,\chi')$. Setting (34) into (22) and (31), we get

$$\psi_l(r,\chi') = \varphi_l(r,\chi') + \frac{1}{2}\varepsilon_l N_l(\chi')\mathrm{P}\int_1^\infty d\rho_l(\cosh\chi)\frac{\tilde{V}_l(\chi)\varphi_l(r,\chi)}{\cosh\chi' - \cosh\chi}, \quad (35)$$

$$N_l(\chi') = \tilde{U}_l(\chi')\left[1 + \mathrm{P}\frac{1}{2}\int_0^\infty d\chi\frac{A_l(\chi)}{\cosh\chi - \cosh\chi'}\right]^{-1}, \quad (36)$$

$$A_l(\chi) = \frac{2}{\pi}\varepsilon_l\left|Q_l(\coth\chi)/F_l^W(\chi)\right|^2\tilde{V}_l(\chi)\tilde{U}_l(\chi). \quad (37)$$

Since the functions $\tilde{V}_l(\chi)$ and $\tilde{U}_l(\chi)$ are differentiable, the principal values of the integrals exist. By virtue of the conditions in (32) and (33), each integral involved is convergent both at the upper and at the lower limits. Thus, the relations (35)-(37) determine the unique solution of Eq.(5) with the boundary condition (6), if only the condition (7) is satisfied.

Using the asymptotic expression (17), the asymptotic behavior of the wave function $\psi_l(r,\chi')$ can be found by representing expression (35) in the form

$$\psi_l(r,\chi') = \frac{|F_l^W(\chi')|}{Q_l(\coth\chi')}\sin\left[r\chi' - \frac{\pi l}{2} + \delta_l^W(\chi')\right] -$$

$$-\varepsilon_l N_l(\chi')\left\{\frac{1}{2\pi i}\int_{-\infty}^\infty d\chi\frac{Q_l(\coth\chi)}{|F_l^W(\chi)|}\frac{\tilde{V}_l(\chi)}{\cosh\chi - \cosh\chi' - i\eta}\exp\left[i(r\chi - \frac{\pi l}{2} + \delta_l^W(\chi))\right] -\right.$$

4 Bound states and Levinson theorem

Suppose that there exists at least one bound state at energy $E' = E_{q'}/m' = \cosh\chi' \geq 0$. The solution of Eq.(26) is then given by

$$\tilde{\psi}_l(\chi', \chi) = -\frac{1}{2}\varepsilon_l N_l(\chi') \mathrm{P} \frac{\tilde{V}_l(\chi)}{\cosh\chi - E'}. \tag{42}$$

Substituting this solution into (31), we arrive at the following equation for the eigenvalues:

$$\Phi_l(\cosh\chi') = \varepsilon_l \left[1 + \mathrm{P}\frac{1}{2}\int\limits_0^\infty d\chi \frac{A_l(\chi)}{\cosh\chi - \cosh\chi'}\right] = 0. \tag{43}$$

From the condition requiring the existence of bound states, it follows that Eq.(43) must have at least one solution. Hence, the function $A_l(\chi)$ must be real-valued which leads to the condition (40). From (41) and (43) it follows that the values of $\varepsilon_l = -1$ correspond to the true bound state of the total interaction whose energy E_t lies in the range

$$0 \leq E' = E_t = \cosh\chi_t < 1, \quad \chi_t = i\kappa_t, \quad 0 < \kappa_t \leq \pi/2. \tag{44}$$

At the same time, Eq.(43) may have solutions at $\varepsilon_l = \pm 1$ for the "spurious" bound states of the non-local separable component of total interaction [1] at energies E_{fk} satisfying the condition

$$E' = E_{fk} = \cosh\chi_{fk} \geq 1, \quad k = \begin{cases} 1, 2, \ldots, \nu_l, & \varepsilon_l = -1, \\ 0, 1, \ldots, \nu_l - 1, & \varepsilon_l = +1. \end{cases} \tag{45}$$

Suppose that there exists a bound state at energy E_t satisfying the condition (44). From (42) and (43), it follows that a bound state at this energy exists, provided that

$$\varepsilon_l = -1, \quad \frac{2}{\pi}\int\limits_0^\infty d\chi \left|\tilde{V}_l(\chi)/F_l^W(\chi)\right|^2 > 1. \tag{46}$$

Obviously, the boundary condition (6) is then satisfied. At the same time, the asymptotic behavior of the wave function determined from (22) and (42) at $\varepsilon_l = -1$ can easily be calculated by applying the residue theorem and performing integration along the boundary of region $0 \leq \mathrm{Im}\chi \leq \pi/2$. As a result, we arrive at

$$\psi_l(r, \chi_t) = N_l(\chi_t)\frac{Q_l(\coth\chi_t)\tilde{V}_l(\chi_t)}{F_l^W(\chi_t)\sinh\chi_t}\exp\left[-\frac{i\pi l}{2} - \kappa_t r\right] \to 0, \quad r \to \infty.$$

Let us now consider the "spurious" bound states at energies E_{fk} satisfying the condition (45). In this case, Eq.(43) can have solutions at $\varepsilon_l = \pm 1$. If such a solution exists, it can be shown with the help of (37a) that the asymptotic behavior of the wave function is given by

$$\psi_l(r, \chi_{fk}) = -\varepsilon_l N_l(\chi_{fk})\frac{Q_l(\coth\chi_{fk})\tilde{V}_l(\chi_{fk})}{|F_l^W(\chi_{fk})|\sinh\chi_{fk}}\cos\left[r\chi_{fk} - \frac{\pi l}{2} + \delta_l^W(\chi_{fk})\right] + O\left(e^{-\pi r/4}\right), \quad r \to \infty.$$

Hence it follows that the wave function asymptotically tends to zero, provided that

$$\tilde{V}_l(\chi_{fk}) = 0. \tag{47}$$

Since the boundary condition (6) is also satisfied, "spurious" bound states correspond to the energies $E_{fk} = \cosh\chi_{fk}$.

Summarizing the above results and using expression (38), we conclude that, if the additional phase-shift curve intersects the straight line $\delta_l^V = \pi k$, where k is an integer, from above, that is conditions (43) and (47) are satisfied, there are "spurious" bound states with the binding energies $E_{fk} = \cosh \chi_{fk}$. By using the estimate (32) and expression (38), we now conclude that $\tan \delta_l^V(\infty) = 0$. For this reason and from the continuity of the function $\delta_l^V(\chi')$, we choose $\delta_l^V(\infty) = 0$. From here, we obtain the Levinson theorem for the case of the superposition of a non-local separable quasipotential and a local one, where the latter does not admit bound states

$$\delta_l^W(0) - \delta_l^W(\infty) \doteq \delta_l^W(0) = 0.$$

This reads

$$\delta_l^V(0) - \delta_l^V(\infty) = \delta_l^V(0) = \pi(\sigma_l + \nu_l), \tag{48}$$

where σ_l is the number of true bound states of the total interaction ($\sigma_l = 0, 1$) with the binding energy satisfying the condition in (44), while ν_l is the number of "spurious" bound states provided the non-local separable component of total interaction satisfies the condition (45).

5 Conclusion

Within the relativistic quasipotential approach to quantum field theory, a method for solving a finite-difference quasipotential equation is developed. It is designed for the case when the total quasipotential simulating the interaction between two relativistic spinless particles of unequal masses is the superposition of a non-local separable quasipotential and a local one, both of them being central. Besides, the local component of the total interaction is supposed to be known and that it can not admit bound states. The proposed approach relies on the possibility of representing the total c.m. energy of two relativistic particles of unequal masses as an expression proportional to the energy of an effective relativistic particle of mass m'. This has permitted us to find an explicit expression for the additional phase-shift, to determine the conditions under which the true and "spurious" bound states may exist, and to generalize the Levinson theorem to the case where the local component of the total interaction does not admit bound states.

Acknowledgments

I am grateful to Yu.S.Vernov, V.V.Andreev, V.N.Kapshai, I.L.Solovtsov and Y.Shnir for interest in this study and for enlightening discussion of the results presented here.

References

[1] K. Chadan,Nuovo Cimento A **XLVII** (1967) 510

[2] M. Bolsterli, J. MacKenzie, Physics**2** (1965) 141

[3] F. Tabakin, Phys. Rev. **177** (1969) 1443

[4] R.L. Mills, J.F. Reading, J. Math. Phys. **10** (1969) 321

[5] R. Barbieri, R. Kogerler, Z. Kunszt, R. Gatto, Nucl. Phys. **B105** (1976) 125

[6] R. McClary, N. Byers, Phys. Rev. D **28** (1983) 1692

[7] A.A. Logunov, A.N. Tavkhelidze, Nuovo Cimento **29** (1963) 380

[8] N.B. Skachkov, I.L. Solovtsov, Yad. Fiz. **30** (1979) 1079; Teor. Mat. Fiz. **41** (1979) 205; Yad. Fiz. **31** (1980) 1332; Teor. Mat. Fiz. **43** (1980) 330

[redacted]

[10] N.A. Boikova, R.N. Faustov, Yu.N. Tyukhtyaev, Yad. Fiz. **64** (2001) 986

[11] V.N. Kapshai, T.A. Alferova, J. Phys. A **32** (1999) 5329

[12] V.G. Kadyshevsky, Nucl.Phys. B **6** (1968) 125

[13] V.G. Kadyshevsky, M.D. Mateev, R.M. Mir-Kasimov, Yad. Fiz. **11** (1970) 692

[14] Yu.D. Chernichenko, Yad. Fiz. **63** (2000) 2068

[15] V.G. Kadyshevsky, R.M. Mir-Kasimov, N.B. Skachkov, Yad. Fiz. **9**, (1969) 212

[16] V.G. Kadyshevsky, R.M. Mir-Kasimov, N.B. Skachkov, Fiz. Elem. Part. At. Nucl., **2** (1972) 635