Relativistic Inverse Problem for a Non-Local Separable Quasipotential

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Abstract

In the framework of the relativistic quasipotential approach to the quantum field theory, a method is developed according to which a non-local separable quasipotential describing interaction between two relativistic spinless particles of unequal masses can be reconstructed by of the phase shift and bound-state energies.

It was proven by Gelfand and Levitan [1, 2], Marchenko [3], and Krein [4, 5] that the inverse problem can, in principle, be solved in the framework of non-relativistic theory. They obtained the linear integral equations in two versions, which served as a basis for a further development of the inverse-problem theory. The most complete survey of this theory was given in the monographs of Chadan and Sabatier [6] and Zakhariev and Suzko [7].

In most of the studies, however, the problem of reconstructing interaction is formulated on the basis of the non-relativistic Schrödinger equation. Therefore, the problem of reconstructing interaction for essentially relativistic systems, in particular, within the relativistic quasipotential approach [8], still remains important.

Within the relativistic quasipotential approach proposed in [9], in this paper the problem is considered for the case where a non-local separable quasipotential simulating interaction between two relativistic spinless particles of unequal masses ($m_1 \neq m_2$) must be reconstructed on the basis of the phase shifts and bound-state energies. The given approach is based on the expression that was found by the present author for the phase shift and which has the form [10] (we use the system of units where $\hbar = c = 1$)

$$\mathrm{tg}\delta_l(\chi') = -\frac{\pi}{2}\mathrm{sh}^{-1}\chi' A_l(\chi')\left[1 + \mathrm{P}\frac{1}{2}\int_0^\infty d\chi \frac{A_l(\chi)}{\mathrm{ch}\chi - \mathrm{ch}\chi'}\right]^{-1}, \tag{1}$$

where the quantity χ' is defined via the relation $E_{q'} = m'\sqrt{1 + (\vec{q}'/m')^2} = m'\mathrm{ch}\chi'$, $m' = \sqrt{m_1 m_2}$, and

$$A_l(\chi') = \frac{2}{\pi}\varepsilon_l Q_l^2(\mathrm{cth}\chi')\left|\tilde{V}_l(\chi')\right|^2, \quad \varepsilon_l = \pm 1. \tag{2}$$

Here, $Q_l(z)$ is a Legendre function of the second kind.

In order to find the quasipotential $V_l(r)$ on the basis of the phase shift $\delta_l(\chi')$, it is necessary to solve the integral equation (1) for the function $A_l(\chi')$. After that, the function $\tilde{V}_l(\chi')$ is determined from Eq.(2). The quasipotential $V_l(r)$ is then reconstructed by performing the relativistic Hankel transformation

$$V_l(r) = \frac{2}{\pi} \int_0^\infty d\chi Q_l(\mathrm{cth}\chi) \tilde{V}_l(\chi) S_l(\chi, r). \tag{3}$$

Here, the function $S_l(\chi, r)$ is a free solution of a finite-difference quasipotential equation in configuration space [11].

In particular, the relativistic Hankel transformation (3) at $l = 0$ reduces to the conventional Fourier transformation

$$V_0(r) = \frac{2}{\pi} \int_0^\infty d\chi \chi \tilde{V}_0(\chi) \sin r\chi.$$

We assume that the phase shift $\delta_l(\chi')$ in Eq.(1) is a function continuous in the sense of Hölder with a positive index and that, for $\chi' \to \infty$, it behaves as

$$\delta_l(\chi') = O\left[(\chi')^{-\gamma}\right], \quad l \geq 0, \quad \gamma > 1. \tag{4}$$

These constraints are necessary and sufficient for the quasipotential to satisfy the condition

$$rV_l(r) \in L_1(0, \infty), \tag{5}$$

which ensures the uniqueness of the inverse-problem solution. We therefore assume that, as χ' increases, the phase shift $\delta_l(\chi')$ intersects the straight lines $\delta_l(\chi') = \pi n$ $(n = 0, 1, 2, ...)$ from above.

Suppose that there exist ν_l $(l \geq 0)$ scattering states at energies satisfying the conditions

$$E'_{Rn} = m'\mathrm{ch}\chi'_{Rn} \geq m', \quad n = 0, 1, ..., \nu_l - 1. \tag{6}$$

We then have

$$\delta_l(0) = \pi\nu_l. \tag{7}$$

In this case $\varepsilon_l = +1$, while the scattering state energies $E'_{Rn} \geq m'$ are found by relation

$$\delta_l(\chi'_{Rn}) = \pi n, \quad n = 0, 1, 2, ..., \nu_l - 1. \tag{8}$$

The integral equation (1) can be reduced to the form

$$A_l(\mathrm{arch}x) g_l^{-1}(x) = 1 + \frac{1}{\pi} \mathrm{P} \int_1^\infty dt \frac{\Psi_l(t) h_l^*(t)}{t - x}, \tag{9}$$

where $x = \mathrm{ch}\chi'$ and where we introduced the following notation:

$$\begin{aligned} \Psi_l(x) &= A_l(\mathrm{arch}x) g_l^{-1}(x) \left[1 + (i\pi/2) g_l(x)(x^2 - 1)^{-1/2}\right], \\ g_l(x) &= -(2/\pi)(x^2 - 1)^{1/2} \mathrm{tg}\Delta_l(x), \\ \Delta_l(x) &= \delta_l(\mathrm{arch}x), \\ h_l(x) &= (\pi/2) g_l(x)(x^2 - 1)^{-1/2} \left[1 - (i\pi/2) g_l(x)(x^2 - 1)^{-1/2}\right]^{-1} = \\ &= -\sin\Delta_l(x) \exp\left[-i\Delta_l(x)\right]. \end{aligned} \tag{10}$$

Using the representation

$$1/(\alpha - io) = i\pi\delta(\alpha) + \mathrm{P}(1/\alpha),$$

Eq.(9) can be recast into the form

$$\Psi_l(x) = 1 + \frac{1}{\pi}\int_1^\infty dt \frac{\Psi_l(t)h_l^*(t)}{t - x - io}. \tag{11}$$

If the function $\Psi_l(x)$ is continuous in the sense of Hölder and if the integral in Eq.(11) converges, then the function

$$H_l(z) = 1 + \frac{1}{\pi}\int_1^\infty dt \frac{\Psi_l(t)h_l^*(t)}{t - z} \tag{12}$$

is analytic in the complex plane of the variable z with the cut from 1 to $+\infty$, and besides the relation

$$\lim_{|z|\to\infty} H_l(z) = 1 \tag{13}$$

holds in all directions. Hence, a solution of the integral Eq.(11) can be represented as

$$\Psi_l(x) = H_l(x_+) = \lim_{\eta\to+0} H_l(x + i\eta), \quad 1 \le x \le \infty. \tag{14}$$

By substituting solution (14) in the expression for the discontinuity of the function $H_l(z)$ over the cut,

$$H_l(x_+) - H_l(x_-) = 2i\Psi_l(x)h_l^*(x) = -2i\sin\Delta_l(x)\exp\left(i\Delta_l(x)\right)\Psi_l(x), \tag{15}$$

we arrive at the homogeneous Riemann–Hilbert equation for the function $H_l(z)$:

$$H_l(x_+)\exp\left(2i\Delta_l(x)\right) - H_l(x_-) = 0, \quad 1 \le x \le \infty. \tag{16}$$

A particular solution satisfying Eq.(16) and the condition in (13) has the form

$$\tilde{H}_l(z) = \exp\left[\omega_l(z)\right], \tag{17}$$

where

$$\omega_l(z) = -\frac{1}{\pi}\int_1^\infty dt \frac{\Delta_l(t)}{t - z}. \tag{18}$$

Besides

$$\lim_{|z|\to\infty} \omega_l(z) = 0, \tag{19}$$

and

$$\omega_l(z) \sim \frac{1}{\pi}\Delta_l(1)\ln|1 - z| \quad \text{for} \quad z \to 1, \tag{20}$$

which holds in all directions, as follows from the assumptions on the behavior of the phase shift and from conditions (4) and (7). Therefore, the function $\tilde{H}_l(z)$ has a zero of order ν_l at the point $z = 1$.

Thus, according to (14), (17), and (18), a particular solution to the non-homogeneous integral equation (11) has the form

$$\tilde{\Psi}_l(x) = \exp\left[\alpha_l(x) - i\Delta_l(x)\right], \tag{21}$$

where

$$\alpha_l(x) = -\frac{1}{\pi}\mathrm{P}\int_1^{\infty} dt \frac{\Delta_l(t)}{t-x}. \tag{22}$$

It should be noted that the function given by (21) is regular at $x = 1$ (it has a zero of order ν_l at this point), continuous in the sense of Hölder with the same index as the phase shift, and limited for $x \to +\infty$. All this is consistent with the a priori assumptions on its properties.

A general solution to the homogeneous equation

$$\Psi_{lo}(x) = \frac{1}{\pi}\int_1^{\infty} dt \frac{\Psi_{lo}(t)h_l^*(t)}{t-x-io} \tag{23}$$

has the form (14), as before, while the function

$$H_{lo}(z) = \frac{1}{\pi}\int_1^{\infty} dt \frac{\Psi_{lo}(t)h_l^*(t)}{t-z} \tag{24}$$

is analytic in the complex plane of the variable z with the cut from 1 to $+\infty$, and besides the relation

$$\lim_{|z|\to\infty} H_{lo}(z) = 0 \tag{25}$$

holds in all directions. Finally, this function satisfies the homogeneous Riemann–Hilbert equation (16). A general solution to this equation will be sought in the form

$$H_{lo}(z) = \sum_{k=1}^{m} A_{k-1}\frac{\exp\left[\omega_l(z)\right]}{(z-1)^k}. \tag{26}$$

Substituting (26) into (16) and requiring that the function $H_{lo}(z)$ is finite at $z = 1$, we obtain $m = \nu_l$. Hence, we have

$$\Psi_{lo}(x) = H_{lo}(x_+) = \sum_{k=1}^{\nu_l} A_{k-1}\frac{\exp\left[\alpha_l(x) - i\Delta_l(x)\right]}{(x-1)^k}. \tag{27}$$

It is obvious that, as in the case of a particular solution, the function in (27) satisfies Eq.(16) and possesses all the required properties.

Therefore, by using notation (10) and transforming the sum as a product, we can recast the general solution to the integral equation (11) in the form

$$A_l(\chi') = -\frac{2}{\pi}\mathrm{sh}\chi' \sin\delta_l(\chi') \exp\left[\alpha_l(\mathrm{ch}\chi')\right] \times \prod_{n=0}^{\nu_l-1}\left[1 - \frac{\mathrm{ch}\chi_{Rn} - 1}{\mathrm{ch}\chi' - 1}\right], \tag{28}$$

where

$$\alpha_l(\mathrm{ch}\chi') = -\frac{1}{\pi}\mathrm{P}\int\limits_0^\infty d\chi \frac{\mathrm{sh}\chi\delta_l(\chi)}{\mathrm{ch}\chi - \mathrm{ch}\chi'}. \tag{29}$$

We note that, in accordance with definition (2), the function $A_l(\chi')$ is of a fixed sign at all values of χ', and for $\varepsilon_l = +1$ it must be positive.

Thus, the solution in (28) is completely determined by the phase shift so far as χ'_{Rn} is also determined by its behavior. Moreover, it follows from expressions (28) and (29) that the function $A_l(\chi')$ is continuous in the sense of Hölder and that, for $\chi' \to +\infty$, it behaves as

$$\mathrm{O}\left[e^{\chi'}(\chi')^{-\gamma}\right], \quad \gamma > 1, \tag{30}$$

provided that the phase shift satisfies condition (4).
This in turn implies that the quasipotential $V_l(r)$ satisfies condition (5).

The case where $\varepsilon_l = -1$ and there are ν_l the scattering states at energies satisfying conditions (6), and n_l bound states whose energies lie the in the range

$$0 \le E'_{ik} = m'\cos\kappa'_{ik} < m', \quad \chi'_{ik} = i\kappa'_{ik}, k = 0, 1, ..., n_l - 1, \tag{31}$$

is considered in the same way.

Besides, by the Levinson theorem, we have

$$\delta_l(0) = \pi(\nu_l + n_l). \tag{32}$$

In accordance with expression (20), the function $\tilde{H}_l(z)$ therefore has a zero of order (ν_l+n_l) at $z = 1$. Further, in the same way as for the case of $\varepsilon_l = +1$, noting that the function $A_l(\chi')$ must now retain a minus sign for all values of χ' for $\varepsilon_l = -1$, we obtain

$$\begin{aligned} A_l(\chi') \;&=\; -\frac{2}{\pi}\mathrm{sh}\chi'\sin\delta_l(\chi')\exp\left[\alpha_l(\mathrm{ch}\chi')\right]\prod_{n=0}^{\nu_l-1}\left[1 - \frac{\mathrm{ch}\chi'_{Rn} - 1}{\mathrm{ch}\chi' - 1}\right] \times \\ &\times \prod_{k=0}^{n_l-1}\left[1 + \frac{1 - \cos\kappa'_{ik}}{\mathrm{ch}\chi' - 1}\right]. \end{aligned} \tag{33}$$

Thus, the function $A_l(\chi')$ is completely determined by the phase shift and bound states, and its sign is opposite to the sign of the phase shift for $\chi' \to +\infty$.

In order to reconstruct the quasipotential $V_l(r)$ by means of the transformation in (3), we can introduce the function

$$\hat{V}_l(\chi') = \prod_{k=0}^{n_l-1}\left[\frac{\mathrm{sh}(\chi'/2) + i\sin(\kappa'_{ik}/2)}{\mathrm{sh}(\chi'/2) - i\sin(\kappa'_{ik}/2)}\right] Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')\big/A_l^{as}(\chi'), \tag{34}$$

where $A_l^{as}(\chi')$ is the asymptotic form of the function

$$\left|Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')\right| = \sqrt{(\pi/2)\varepsilon_l A_l(\chi')}$$

for $|\chi'| \to \infty$.

The function $\hat{V}_l(\chi')$ is analytic in the region $\mathrm{Im}\chi' > 0$, it is continuous for $\mathrm{Im}\chi' \geq 0$ and satisfies the condition

$$\hat{V}_l(\chi') = 1 + o(1), \quad |\chi'| \to \infty, \tag{35}$$

provided that is satisfied condition (5). Besides, the function $\hat{V}_l(\chi')$ vanishes nowhere for $\mathrm{Im}\chi' > 0$. Hence, the function $\ln \hat{V}_l(\chi')$ is analytic in the region $\mathrm{Im}\chi' > 0$ and tends to zero at infinity because of the estimate in (35). Therefore, we can apply the integral Hilbert transformation to the real and imaginary parts of the function $\ln \hat{V}_l(\chi')$ setting

$$Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi') = \left|Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')\right| \exp\left[i\Phi_l(\chi')\right]. \tag{36}$$

We then obtain

$$\mathrm{Im}\ln \hat{V}_l(\chi') = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} d\mathrm{sh}(\chi/2)\frac{\mathrm{Re}\ln \hat{V}_l(\chi)}{\mathrm{sh}(\chi/2) - \mathrm{sh}(\chi'/2)} = \tag{37}$$

$$= i\ln\left[\left|Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')\right| / A_l^{as}(\chi')\right] - \frac{1}{\pi}\int_{-\infty}^{\infty} d\mathrm{sh}(\chi/2)\frac{\ln\left[\left|Q_l(\mathrm{cth}\chi)\tilde{V}_l(\chi)\right| / A_l^{as}(\chi)\right]}{\mathrm{sh}(\chi/2) - \mathrm{sh}(\chi'/2) - io}.$$

Combining (37) with the expression for

$$\mathrm{ReLen}\hat{V}_l(\chi') = \ln\left[\left|Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')\right| / A_l^{as}(\chi')\right], \tag{38}$$

we now obtain the formula

$$\ln \hat{V}_l(\chi') = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mathrm{sh}(\chi/2)\frac{\ln\left[(\pi/2)\varepsilon_l A_l(\chi)/\left(A_l^{as}(\chi)\right)^2\right]}{\mathrm{sh}(\chi/2) - \mathrm{sh}(\chi'/2)}, \tag{39}$$

which is valid in the region $\mathrm{Im}\chi' > 0$. Finally, from expressions (34) and (39), it follows that

$$Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi') = A_l^{as}(\chi')\prod_{k=0}^{n_l-1}\left[\frac{\mathrm{sh}(\chi'/2) - i\sin(\kappa'_{lk}/2)}{\mathrm{sh}(\chi'/2) + i\sin(\kappa'_{lk}/2)}\right] \times \tag{40}$$

$$\times \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mathrm{sh}(\chi/2)\frac{\ln\left[(\pi/2)\varepsilon_l A_l(\chi)/\left(A_l^{as}(\chi)\right)^2\right]}{\mathrm{sh}(\chi/2) - \mathrm{sh}(\chi'/2)}\right\},$$

which is valid for $\mathrm{Im}\chi' > 0$.

Thus, a solution to the inverse problem exists and is completely determined as soon as the function $A_l(\chi')$ is found by the phase shift and bound-state energies for $l \geq 0$.

To summarize, we note that the method proposed here to reconstruct a non-local separable quasipotential simulating interaction between two relativistic spinless particles of unequal masses actually reduces to a one-body problem. This is thanks to the possibility of representing, within the relativistic quasipotential approach to quantum field theory, the total c.m. energy of two relativistic particles of unequal masses as an expression proportional to the energy of an effective relativistic particle of mass m'.

Acknowledgments

I am grateful to Yu.S.Vernov, V.I.Savrin, and A.M.Shirokov for a permanent interest in this study and for enlightening discussions on the results presented here.

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