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Summation of threshold singularities in quantum chromodynamics

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The new relativistic Coulomb-like threshold resummation S- and L-factors in quantum chromodynamics are received. Consideration is conducted within the framework of quasipotential approach in quantum field theory formulated in the relativistic configurational representation in the case of two particles of unequal masses. We also suggest a new model expression for $R(s)$ in which threshold singularities are summarized into a main potential contribution.

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1. Introduction

At the description of quark-antiquark systems close to threshold we can not cut off the perturbative series even if the expansion parameter, the QCD coupling constant α_S, is small [1]. The problem is well known from QED [2]. The reason consist in that the real expansion parameter in the threshold region is α/v , where $v = \sqrt{1 - 4m^2/s}$ is a quark velocity, and m is a quark mass. Obviously, it becomes to be singular, when the velocity $v \to 0$. To obtain meaningful result these threshold singularities of the form $(\alpha/v)^n$ have to be summarized. In the nonrelativistic of case for the Coulomb interaction

$$V(r) = -\alpha/r \tag{1}$$

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this resummation is realized the known S-factor Gamov Sommerfeld Sakharov [3–5]

$$S_{\rm nr} = \frac{X_{\rm nr}}{1-\exp(-X_{\rm nr})}\,, \qquad X_{\rm nr} = \frac{\pi\,\alpha}{v_{\rm nr}}\,, \tag{2}$$

which is related to the wave function of the continuous spectrum at the origin by $|\psi(0)|^2$. Here $2\,v_{\rm nr}$ is the relative velocity of two nonrelativistic particles. The corresponding nonrelativistic expression can also be obtained for higher ℓ states (see, e.g., [6]).

In the relativistic theory the nonrelativistic approximation needs to be modified. For the first time the relativistic modification of the S-factor (2) in QCD in the case of two particles of equal masses ($m_1 = m_2 = m$) was executed in [7] (see also [8]) and it consisted in the change $v_{\rm nr} \to v$. This factor was used for the description of effects close to the threshold of pair production in the processes $e^+e^- \to t\bar{t}$ and $e^+e^- \to W^+W^-$. Just the same form of the S-factor for the interaction of two particles of equal masses was later suggested in [9]. Another form of the relativistic generalization of the S-factor also in the case of two particles of equal masses was obtained in [10]. The relativistic S-factor for two particles of arbitrary masses ($m_1 \neq m_2$) was presented in [11]. This factor was derived within the framework of relativistic quantum mechanics on the basis of the Schrödinger equation.

The new method to relativistic generalization of the S-factor in the case of two particles of equal masses was developed by Milton and Solovtsov in [12]. Their the method is based on the relativistic quasipotential (RQP) approach proposed by Logunov and Tavkhelidze [13] in the form suggested by Kadyshevsky [14]. In the method developed by them, the possibility of transformation of quasipotential (QP) equation from momentum space into relativistic configurational representation in the case of two particles of equal masses (see [15]) has been used also. Moreover, it is important the potential (1) that used by them possesses the QCD-like behaviour (see [16]). The solution containing arbitrary functions of r with period

i, the so-called the i-periodic constants, with the same potential was investigated in [17]. However, the using of such solution is suitable for the spectral problems only. Other forms of the QP equation with the Coulomb potential were considered in [18].

Thus, in [12] a new step to application of the quasipotential approach in QCD was made. This approach gives the following expression for the relativistic S-factor:

$$S(\chi) = \frac{X(\chi)}{1-\exp\left[-X(\chi)\right]}, \qquad X(\chi) = \frac{\pi\,\alpha}{\sinh\chi}, \tag{3}$$

where χ is the rapidity related to the total c. m. energy of interacting particles, $\sqrt{s}$, by $2m\cosh\chi = \sqrt{s}$. The function $X(\chi)$ in Eq. (3) can be expressed in terms of v as $X(\chi) = \pi\,\alpha\sqrt{1-v^2}/v$. The method proposed by them in [12] has been generalized in [19] successfully to get the following expression for the relativistic L-factor ($\ell \geq 0$) in the case of two particles of equal masses:

$$L(\chi) = \prod_{n=1}^{\ell}\left[1+\left(\frac{\alpha}{2\,n\,\sinh\chi}\right)^2\right]\frac{X(\chi)}{1-\exp[-X(\chi)]}, \tag{4}$$

where the function $X(\chi)$ is determined in (3). Applications of the factor (3) for describing some hadronic processes can be found in [20–22]. Recently, the relativistic S-factor (3) has been applied in [23] to reanalyze the mass limits obtained for magnetic monopoles which might have been produced at the Fermilab Tevatron.

We should like to remind that the resummation factor appears in the parametrization of the imaginary part of the quark current correlator, the Drell ratio $R(s)$, which can be approximated in terms of the Bethe-Salpeter (BS) amplitude of two charged particles $\chi_{\mathrm{BS}}(x)$ at $x = 0$ (see [24]). The nonrelativistic replacement of this amplitude by the wave function, which obeys the Schrödinger equation with the Coulomb potential (1), leads to formula (2) with a substitution $\alpha \to 4\alpha_{\mathrm{S}}/3$ for QCD. The possibility of using the QP approach for our task is based on the fact that the BS amplitude, which parameterizes the physical quantity $R(s)$, is taken at $x = 0$; therefore, in particular, at the relative time $\tau = 0$. Thus, the QP wave function in the momentum space is defined as the BS amplitude at $\tau = 0$, and, therefore, $R(s)$ can be expressed through the QP wave function in the momentum space, $\Psi_q(\mathbf{p})$, by using the relation

$$\chi_{\rm BS}(x=0) = \frac{1}{(2\pi)^3} \int d\Omega_p \, \Psi_q(\mathbf{p}) \,, \tag{5}$$

where $d\Omega_p = (m\, d\mathbf{p})/E_p$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid $E_p^2 - \mathbf{p}^2 = m^2$.

The purpose of this paper is to generalize the previous study started in [12] to the case of the interaction of two particles of unequal masses ($m_1 \neq m_2$). The paper is organized as follows. In the next section, we present the formalism of the RQP approach in quantum field theory formulated in the relativistic configuration representation for the interaction of two relativistic particles of unequal masses [25]. In Sec. III, within the framework of this approach we derive the new relativistic S- and L-factors. In Sec. IV we analyze their behavior in the following cases: the nonrelativistic and relativistic cases, the case of equal masses, the case of one particle being at rest, and the ultrarelativistic case. Also, we study the behavior of the function $R(s)$ that is expressed in terms of these factors in the case of vector current for two quarks of unequal masses. Summarizing comments are given in Sec. V.

2. The integral form of quasipotential equation in the case of two particles of unequal masses

The basis of our consideration is QP equation into the momentum space constructed in [25] for the RQP wave function $\Psi_{q'}(\mathbf{p}')$ of two relativistic particles of unequal masses. This equation is given by (in the following we will use the system of units $c = \hbar = 1$)

$$(2E_{q'} - 2E_{p'})\, \Psi_{q'}(\mathbf{p}') = \frac{2\,\mu}{m'\,(2\pi)^3} \int d\Omega_{k'}\, \widetilde{V}\,(\mathbf{p}'\,, \mathbf{k}';\, E_{q'})\, \Psi_{q'}(\mathbf{k}')\,, \qquad (6)$$

where

$$d\Omega_{k'} = \frac{m'\, d\mathbf{k}'}{E_{k'}}$$

is the relativistic three-dimensional volume element in the Lobachevsky space, $E_{k'} = \sqrt{m'^{\,2} + \mathbf{k}'^{\,2}}$, $m' = \sqrt{m_1\, m_2}$, and $\mu = m_1\, m_2/(m_1 + m_2)$ is the usual reduced mass.

Eq. (6) represents a relativistic generalization of the Lippmann-Schwinger equation in the spirit of the Lobachevsky geometry, which is realized on the upper half of the mass hyperboloid $E^2_{k'} - \mathbf{k}'^{\,2} = m'^{\,2}$. This equation describes the scattering over the quasipotential $\widetilde{V}\,(\mathbf{p}', \mathbf{k}';\, E_{q'})$ of an effective relativistic particle having mass m' and a relative 3-momentum $\mathbf{k}'$, emerging instead of the system of two particles and carrying the total c. m. energy of the interacting particles, $\sqrt{s}$, proportional to the energy $E_{k'}$ of one effective relativistic particle of mass m' (see [25, 26]):

$$\sqrt{s} = \sqrt{m_1{}^2 + \mathbf{k}^2} + \sqrt{m_2{}^2 + \mathbf{k}^2} = \frac{m'}{\mu} E_{k'}, \qquad E_{k'} = \sqrt{m'^{\,2} + \mathbf{k}'^{\,2}}. \qquad (7)$$

The proper Lorentz transformations means a translation in the Lobachevsky space. The role of the plane waves corresponding to these translations are played by the following functions:

$$\xi(\mathbf{p}'\,, \mathbf{r}) = \left(\frac{E_{p'} - \mathbf{p}' \cdot \mathbf{n}}{m'} \right)^{-1 - i r\, m'}, \qquad (8)$$

where the module of the radius-vector, $\mathbf{r}$, ($\mathbf{r} = r\,\mathbf{n}\,, |\mathbf{n}| = 1$) is a relativistic invariant [26]. These functions correspond to the principal series of unitary representations of the Lorentz group and in the nonrelativistic limit ($p' \ll 1,\ r \gg 1$) $\xi(\mathbf{p}'\,, \mathbf{r}) \to \exp(i\mathbf{p}' \cdot \mathbf{r})$. The functions (8) obey the following conditions of completeness and orthogonality [26]:

$$\frac{1}{(2\pi)^3}\int d\Omega_{p'}\,\xi(\mathbf{p}',\mathbf{r})\,\xi^*(\mathbf{p}',\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'),$$
$$\frac{1}{(2\pi)^3}\int d\mathbf{r}\,\xi(\mathbf{p}',\mathbf{r})\,\xi^*(\mathbf{q}',\mathbf{r}) = \delta(\mathbf{p}'(-)\mathbf{q}'), \tag{9}$$

where $\delta(\mathbf{p}'(-)\mathbf{q}') = \sqrt{1+\mathbf{p}'^2/m'^2}\,\delta(\mathbf{p}'-\mathbf{q}')$ is the relativistic δ-function in momentum-space. Moreover, these the functions satisfy the equation in terms of finite differences

$$\left(2E_{p'} - \hat{H}_0\right)\xi(\mathbf{p}',\mathbf{r}) = 0\,. \tag{10}$$

Here

$$\hat{H}_0 = 2\,m'\left[\cosh\left(i\lambda'\frac{\partial}{\partial r}\right) + \frac{i\lambda'}{r}\sinh\left(i\lambda'\frac{\partial}{\partial r}\right) - \frac{\lambda'^2\,\Delta_{\theta,\varphi}}{2r^2}\exp\left(i\lambda'\frac{\partial}{\partial r}\right)\right] \tag{11}$$

is the operator of the free Hamiltonian, while $\Delta_{\theta,\varphi}$ is its the angular part and $\lambda' = 1/m'$ is the Compton wavelengs associated with the effective relativistic particle of mass m'.

We note that Eq. (6) differs from of the QP equation considered in [27] by means of introduction into it of the relativistic reduced mass. However, in [27] was shown that it is possible to use the different expressions for the relativistic reduced mass by means of the choice of functional relationship between the relative 3-momentum $\mathbf{k}$ and the relativistic relative velocity of interacting particles, $\mathbf{v}$, connected, as is well known, with their the total c. m. energy of interacting particles, $\sqrt{s}$, by relation (see, for instance, [10, 11])

$$|\mathbf{v}| = 2\sqrt{\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}}\left(1+\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}\right)^{-1}. \tag{12}$$

In particular, if the dependence between the energy of the relative motion and the relativistic relative velocity $\mathbf{v}$ is given by expression (see [25, 26])

$$\frac{\mathbf{k}'^2}{2\mu} = \mu\left(\frac{1}{\sqrt{1-|\mathbf{v}|^2}} - 1\right), \tag{13}$$

this together with relation (12) leads to the expression (7). Such the choice of functional relationship has allowed to enter the concept of an effective relativistic particle, emerging instead of the system of two particles and having mass m', the relative 3-momentum $\mathbf{k}'$ and carrying the total c. m. energy of interacting particles, $\sqrt{s}$. Notice that the relative 3-momentum $\mathbf{k}'$ of an effective relativistic particle, according to the expression (13), is invariant of the Loretz transformations.

The QP wave functions in the momentum space and relativistic configurational representation [25, 26] are related as follows:

$$\begin{aligned} \psi_{q'}(\mathbf{r}) &= \frac{1}{(2\pi)^3} \int d\Omega_{p'}\, \xi(\mathbf{p}', \mathbf{r})\, \Psi_{q'}(\mathbf{p}')\,, \\ \Psi_{q'}(\mathbf{p}') &= \int d\mathbf{r}\, \xi^*(\mathbf{p}', \mathbf{r})\, \psi_{q'}(\mathbf{r}). \end{aligned} \tag{14}$$

For a spherically symmetric potential the application of transformations (14) (Shapiro transformations or ξ-transformations) to Eq. (6) leads to the equation, which is the integral form of the relativistic Schrödinger equation in the configurational representation:

$$\begin{aligned} &\frac{1}{(2\pi)^3} \int d\Omega_{p'}\, (2E_{q'} - 2E_{p'})\, \xi(\mathbf{p}', \mathbf{r}) \int d\mathbf{r}'\, \xi^*(\mathbf{p}', \mathbf{r}')\, \psi_{q'}(\mathbf{r}') \\ &= \frac{2\mu}{m'} V(r\,; E_{q'})\, \psi_{q'}(\mathbf{r})\,, \end{aligned} \tag{15}$$

where the right-hand side is already local in the configuration representation and the transform of the potential, $V(r\,; E_{q'})$, is given in terms of the same relativistic plane waves.

We note that the using of relations (14) and Eq. (10) allows us to express the left-hand side of Eq. (15) in terms finite differences

$$\left(2E_{q'} - \hat{H}_0\right) \psi_{q'}(\mathbf{r}) = \frac{2\,\mu}{m'} V(r\,; E_{q'})\, \psi_{q'}(\mathbf{r})\,. \tag{16}$$

Solutions of this equation, in principle, can contain arbitrary functions of r with period i, the so-called the i-periodic constants, which appear in the solutions due to the finite difference nature of the Hamiltonian (11). For some problems, such as defining the bound state spectrum, this i-periodic constant is not important. However, for the purpose of extracting resummation factors one must develop a method which avoids this ambiguity. For this instead of Eq. (16) we will to use Eq. (15).

The integral Eq. (15) can be reduced to the form

$$\begin{aligned}&\frac{1}{(2\pi)^3}\int d\Omega_p\,(2E_q-2E_p)\,\xi(\mathbf{p}\,,\boldsymbol{\rho})\int d\boldsymbol{\rho}'\,\xi^*(\mathbf{p}\,,\boldsymbol{\rho}')\,\psi_q(\boldsymbol{\rho}')\\&=\frac{2\,\mu}{m'}\,V(\rho\,;E_q)\,\psi_q(\boldsymbol{\rho})\,,\end{aligned} \tag{17}$$

where we introduced the following notation:

$$\begin{aligned}&\mathbf{q}'=m'\mathbf{q}\,,\mathbf{p}'=m'\mathbf{p}\,,\mathbf{q}=\sinh(\chi_q)\,\mathbf{n}_q\,,\mathbf{p}=\sinh(\chi_p)\,\mathbf{n}_p\,,\\&|\mathbf{n}_q|=|\mathbf{n}_p|=1\,,\\&\boldsymbol{\rho}=m'\mathbf{r}\,,\boldsymbol{\rho}'=m'\mathbf{r}'\,,\rho=|\boldsymbol{\rho}|\,,\rho'=|\boldsymbol{\rho}'|\,,r=|\mathbf{r}|\,,r'=|\mathbf{r}'|\,,\\&d\mathbf{r}'=m'^{-3}d\boldsymbol{\rho}'\,,\\&d\Omega_{p'}=m'^3d\Omega_p\,,d\Omega_p=\frac{d\mathbf{p}}{E_p}\,,E_{q'}=m'E_q\,,E_{p'}=m'E_p\,,E_q=\sqrt{1+\mathbf{q}^2}\,,\\&E_p=\sqrt{1+\mathbf{p}^2}\,,\\&V(r\,;E_{q'})=V(\rho/m'\,;E_{q'})\equiv m'V(\rho\,;E_q)\,,\\&\xi(\mathbf{p}'\,,\mathbf{r})=(E_p-\mathbf{p}\cdot\mathbf{n})^{-1-i\rho}\equiv\xi(\mathbf{p}\,,\boldsymbol{\rho})\,,\\&\psi_{q'}(\mathbf{r})=\psi_{m'q}(\boldsymbol{\rho}/m')\equiv\psi_q(\boldsymbol{\rho})\,,\Psi_{q'}(\mathbf{p}')\equiv m'^{-3}\Psi_q(\mathbf{p})\,.\end{aligned} \tag{18}$$

By using the expansions

$$\begin{aligned}\xi(\mathbf{p}\,,\boldsymbol{\rho})&=\sum_{\ell=0}^{\infty}(2\ell+1)\,i^\ell\,p_\ell(\rho\,,\cosh\chi_p)\,P_\ell\!\left(\frac{\mathbf{p}\cdot\boldsymbol{\rho}}{p\,\rho}\right),\\\psi_q(\boldsymbol{\rho})&=\sum_{\ell=0}^{\infty}(2\ell+1)\,i^\ell\,\frac{\varphi_\ell(\rho\,,\chi_q)}{\rho}\,P_\ell\!\left(\frac{\mathbf{q}\cdot\boldsymbol{\rho}}{q\,\rho}\right),\end{aligned} \tag{19}$$

and also formula [15]

$$p_\ell(\rho\,,\cosh\chi) = \frac{(-1)^\ell\,(\sinh\chi)^\ell}{\rho^{(\ell+1)}}\left(\frac{d}{d\cosh\chi}\right)^\ell\left(\frac{\sin\rho\chi}{\sinh\chi}\right),$$

Eq. (17) transformed to the form

$$\frac{2}{\pi}\int\limits_0^\infty d\chi'\frac{(\sinh\chi')^{2\ell+2}\,(-1)^{\ell+1}}{\rho^{(\ell+1)}}\,(2\cosh\chi-2\cosh\chi')\left(\frac{d}{d\cosh\chi'}\right)^\ell\left(\frac{\sin\rho\chi'}{\sinh\chi'}\right)\times$$

$$\times\left(\frac{d}{d\cosh\chi'}\right)^\ell\frac{1}{\sinh\chi'}\int\limits_0^\infty d\rho'\frac{\rho'\,\sin\rho'\chi'}{(-\rho')^{(\ell+1)}}\,\varphi_\ell(\rho'\,,\chi) = \frac{2\,\mu}{m'}\,\frac{V(\rho\,;E_q)\,\varphi_\ell(\rho\,,\chi)}{\rho}\,. \tag{20}$$

Here $P^\nu_\mu(z)$ is a Legendre function of the first kind, and the function

$$p_\ell(\rho\,,\cosh\chi) = \frac{(-1)^{\ell+1}}{\rho}\sqrt{\frac{\pi}{2\sinh\chi}}\,(-\rho)^{(\ell+1)}\,P^{-1/2-\ell}_{-1/2+i\rho}(\cosh\chi) \tag{21}$$

is the solution of Eq. (16) in the case when the interaction is switched off, $V(r\,;E_{q'})\equiv 0$; χ' and χ are the rapidities which are related to E_p and E_q as $E_p=\cosh\chi'\,, E_q=\cosh\chi$, and the function

$$(-\rho)^{(\ell+1)} = i^{\ell+1}\frac{\Gamma(\ell+1+i\rho)}{\Gamma(i\rho)} \tag{22}$$

is the generalized power [15] where $\Gamma(z)$ is the gamma-function.

Thus, Eq. (20) differs from the corresponding equation in the case of two particles of equal masses (see [28]) only by the factor $2\mu/m'$ turning into 1 at $m_1=m_2$.

3. Relativistic threshold resummation S- and L-factors

For the first time, the solution of Eq. (20) in the case of the interaction of two relativistic particles of equal masses at $\ell=0$, not containing the i-periodic constants, was received in [12]. This approach leads to the relativistic S-factor (3).

We note that the applying of ξ-transformation (14) to the Coulomb interaction (1) gives the potential in momentum space (see [12])

$$V(\Delta) \sim \frac{1}{\chi_\Delta \sinh \chi_\Delta},$$

where the relative rapidity χ_Δ corresponds to $\mathbf{\Delta} = \mathbf{p}'(-)\mathbf{k}'$ and is defined in terms of the square of the momentum transfer by $Q^2 = -(p' - k')^2 = 2(\cosh \chi_\Delta - 1)$. For large Q^2 the potential $V(\Delta)$ behaves as $(Q^2 \ln Q^2)^{-1}$, which reproduces the principal behaviour of the QCD potential proportional to $\bar{\alpha}_S(Q^2)/Q^2$ with $\bar{\alpha}_S(Q^2)$ being the QCD running coupling. This property of the potential (1), its QCD-like behaviour, was noted in [16].

To solve the RQP equation (20) with the potential (1), we use the method developed in [12] (see also [28, 29]). Generalising this method, we will seek a solution of RQP equation (20) with the potential (1) in the form

$$\varphi_\ell(\rho\,,\chi) = \frac{(-\rho)^{(\ell+1)}}{\rho} \int\limits_{\alpha_-}^{\alpha_+} d\zeta\, e^{i\rho\zeta}\, R_\ell(\zeta\,,\chi)\,, \tag{23}$$

where the ζ-integration is performed in the complex plane over a contour with end points α_- and α_+ (see Fig. 1). Substituting (23) into (20) and taking into account that

$$\frac{1}{i\pi} \int\limits_0^\infty d\rho' \sin(\rho'\chi')\, e^{i\rho'\zeta} = \frac{1}{i\pi} \frac{\chi'}{\chi'^2 - \zeta^2}\,,$$

we arrive at the equation

$$(-1)^\ell \int\limits_{\alpha_-}^{\alpha_+} d\zeta\, R_\ell(\zeta\,,\chi) \left(\frac{d}{d\cosh\zeta}\right)^\ell \Bigg[(\sinh\zeta)^{2\ell+1}\,(2\cosh\chi - 2\cosh\zeta) \times$$

$$\times \left(\frac{d}{d\cosh\zeta}\right)^\ell \left(\frac{e^{i\rho\zeta}}{\sinh\zeta}\right)\Bigg] = -\frac{2\,\alpha\,\mu}{m'\rho} \prod_{n=1}^{\ell} (\rho^2 + n^2) \int\limits_{\alpha_-}^{\alpha_+} d\zeta\, e^{i\rho\zeta}\, R_\ell(\zeta\,,\chi)\,. \tag{24}$$

It should be noted that solutions of this equation, and hence of Eq. (20), do not contain any more the i-periodic constants which appear in the solutions of Eq. (16) due to the finite difference nature of the Hamiltonian (11).

Eq. (24) at $\ell = 0$ which, when we integrate by parts, leads to the following equation

$$\frac{d}{d\zeta}\Big[\,(\cosh\chi - \cosh\zeta)\,R_0(\zeta\,,\chi)\Big] - \frac{i\alpha\,\mu}{m'}\,R_0(\zeta\,,\chi) = 0\,, \tag{25}$$

with the boundary condition

$$e^{i\rho\zeta}\,(\cosh\chi - \cosh\zeta)\,R_0(\zeta\,,\chi)\Big|_{\zeta=\alpha_-}^{\zeta=\alpha_+} = 0\,. \tag{26}$$

As a result the solution of Eq. (25) is

$$R_0(\zeta\,,\chi) = C_0(\chi)\frac{e^{\zeta}}{(e^{\zeta}-e^{\chi})^2}\left[\frac{e^{\zeta}-e^{-\chi}}{e^{\zeta}-e^{\chi}}\right]^{-1+iA}\,, \tag{27}$$

where $A = \alpha\,\mu/(m'\sinh\chi)$, and $C_0(\chi)$ is an arbitrary function of χ. The branch points of the function (27) are $\pm\chi + 2\pi ni$ (see Fig. 1).

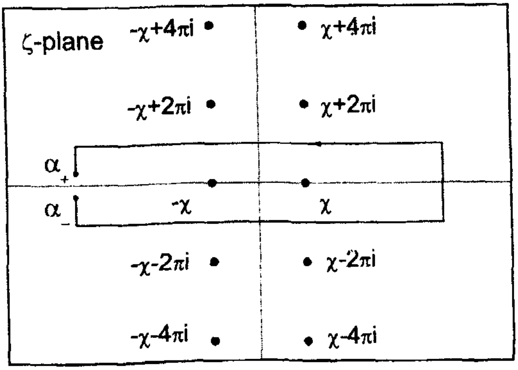

Fig. 1. Contour of integration in Eq. (23) and singularities of the function (27) in the complex ζ-plane.

The contour of integration must not intersect cuts which we take from $-\infty + 2\pi ni$ to $\pm\chi + 2\pi ni$. In the case when the interaction vanishes,

$\alpha \to 0$, the solution $\varphi_\ell(\rho, \chi)$ should reproduce the known free wave function $\rho p_\ell(\rho, \cosh\chi) \xrightarrow[\rho\to\infty]{} \sin(\rho\chi - \pi\ell/2)/\sinh\chi$. Taking into account these remarks and the boundary condition (26), we take $\alpha_- = -R - i\varepsilon, \alpha_+ = -R + i\varepsilon$ with $R \to +\infty, \varepsilon \to +0$. The vertical part of the contour to the right is given by $\mathrm{Re}\,\zeta = +R$. It is also convenient for finding a connection to an integral representation of the hypergeometric function to take the horizontal parts of the contour to be characterized by $\mathrm{Im}\,\zeta = \pm\pi$ (see Fig. 1).

Substituting the solution (27) into (23) at $\ell = 0$, we obtain the following expression for the RQP partial wave function $\varphi_0(\rho, \chi)$:

$$\varphi_0(\rho, \chi) = C_0(\chi)\frac{\rho}{\rho^{(1)}}\int\limits_{\alpha_-}^{\alpha_+} d\zeta \frac{e^{(i\rho+1)\zeta}}{(e^\zeta - e^\chi)^2}\left[\frac{e^\zeta - e^{-\chi}}{e^\zeta - e^\chi}\right]^{-1+iA}, \qquad A = \frac{\alpha\,\mu}{m'\sinh\chi}. \tag{28}$$

Performing in Eq. (28) ζ-integration in the complex plane along a contour with end points α_- and α_+ (in the same way as in [12, 28, 29]) we obtain the resulting solution which does not contain the i-periodic constant in the form

$$\varphi_0(\rho, \chi) = C_0(\chi)\frac{2\,\rho\,\sinh(\pi\,\rho)}{\rho^{(1)}}\int\limits_{-\infty}^{\infty} dx \frac{e^{(i\rho+1)x}}{(e^x + e^\chi)^2}\left[\frac{e^x + e^{-\chi}}{e^x + e^\chi}\right]^{-1+iA}. \tag{29}$$

The solution (29) can also be represented in terms of hypergeometrical function as

$$\varphi_0(\rho, \chi) = -N_0(\chi)(-\rho)^{(1)}\, e^{i\rho\chi + iA\chi}\, F\left(1 - iA, 1 - i\rho; 2; 1 - e^{-2\chi}\right). \tag{30}$$

The normalization constant $N_0(\chi)$ in Eq. (30) can be obtained (also as in [12]) from the condition

$$\lim_{\alpha\to 0}\varphi_\ell(\rho, \chi) = \rho\, p_\ell(\rho, \cosh\chi) \xrightarrow[\rho\to\infty]{} \frac{\sin(\rho\chi - \pi\,\ell/2)}{\sinh\chi} \tag{31}$$

at $\ell = 0$, where the function $p_\ell(\rho, \cosh\chi)$ is determined in (21).

We should like to remind that the Bethe-Salpeter amplitude $\chi_{\rm BS}(x=0)$ is associated with the RQP wave function in the momentum space, $\Psi_q(\mathbf{p})$, by the relation (5). Taking into account the transformations (14) and notations (18), the relationship of the Bethe Salpeter amplitude with the RQP wave function, $\psi_q(\boldsymbol{\rho})$, is

$$\chi_{\rm BS}(x=0) = \psi_q(\boldsymbol{\rho})|_{\rho=i} \,.$$

The generalized power (22) in the solution (23) vanishes at $\rho = i$ for all $\ell \neq 0$. Thus, the expansion (19) for the wave function $\psi_q(\boldsymbol{\rho})$ contains only s-wave ($\ell = 0$). Hence, by using relations (30) and (31) we can calculate $|\psi_q(i)|^2$, which leads to the following expression for the relativistic S-factor in the case of two particles of unequal masses:

$$\begin{aligned} S_{\rm uneq}(\chi) &= \lim_{\rho\to i}\left|\frac{\varphi_0(\rho\,,\chi)}{\rho}\right|^2 = \frac{X_{\rm uneq}(\chi)}{1-\exp\left[-X_{\rm uneq}(\chi)\right]}\,, \\ X_{\rm uneq}(\chi) &= \frac{2\,\pi\,\alpha\,\mu}{m'\,\sinh\chi}\,, \end{aligned} \tag{32}$$

where χ is the rapidity which is related to the total c. m. energy, $\sqrt{s}$, as $(m'^2/\mu)\cosh\chi = \sqrt{s}$.

The L-factor in the nonrelativistic case is defined by derivative of the order ℓ of the the wave function at $r = 0$. In the relativistic case, instead of the derivative, one has to use its finite difference analog [15]:

$$\Delta^* = \frac{1}{i}\left[\exp\left(i\frac{\partial}{\partial\rho}\right) - 1\right]. \tag{33}$$

Thus, the relativistic L-factor is connected, as one can expect, with the RQP partial wave function $\varphi_\ell(\rho\,,\chi)$ as follows:

$$L_{\rm uneq}(\chi) = \lim_{\rho\to i}\left|\frac{\Gamma(2\,\ell+2)}{(2\,\sinh\,\chi)^\ell\,\Gamma^2(\ell+1)}(\Delta^*)^\ell\left[\frac{\varphi_\ell(\rho\,,\chi)}{\rho}\right]\right|^2 . \tag{34}$$

The resulting solution Eq. (24) at arbitrary $\ell \geq 0$ which does not contain the i-periodic constant, can also be represented in terms of hypergeometrical function by

$$\begin{aligned}\varphi_\ell(\rho\,,\chi) = {}& N_\ell(\chi)(-\rho)^{(\ell+1)}\, e^{i\rho\chi+iA\chi+i\pi(\ell+1)}\times \\ &\times F\left(\ell+1-iA\,,\ell+1-i\rho;2\ell+2\,;1-e^{-2\chi}\right)\,.\end{aligned} \tag{35}$$

The normalization constant $N_\ell(\chi)$ in Eq. (35) can be obtained (also as in case s-wave, $\ell=0$) from the condition (31).

By using Eqs. (31) and (33)–(35), we finally find the following expression for the relativistic L-factor in the case of two particles of unequal masses:

$$L_{\,\mathrm{uneq}}(\chi) = \prod_{n=1}^{\ell}\left[1+\left(\frac{\alpha\,\mu}{m'\,n\,\sinh\chi}\right)^2\right] S_{\,\mathrm{uneq}}(\chi)\,. \tag{36}$$

The functions $\sinh\chi$ and $X_{\,\mathrm{uneq}}(\chi)$ in Eqs. (32) and (36) can be expressed in terms of the "velocity" u determined by the relation

$$u = \sqrt{1-\frac{4\,m'^2}{s-(m_1-m_2)^2}}\,, \tag{37}$$

in the form

$$\sinh\chi = \frac{2\,\mu\,u}{m'\,\sqrt{1-u^2}}\,, \qquad X_{\,\mathrm{uneq}}(\chi) = \frac{\pi\,\alpha\,\sqrt{1-u^2}}{u}\,. \tag{38}$$

We note that the square of relative 3-momentum $\mathbf{k}'$ for an effective relativistic particle, having mass m', the total c. m. energy of interacting particles, $\sqrt{s}$, and emerging instead of the system of two particles, is defined by formula (7) and connected with the relative relativistic velocity of interacting particles, $\mathbf{v}$, by the expression (13). In turn the relative velocity $\mathbf{v}$ of interacting particles is expressed through their the total c. m. energy of interacting particles, $\sqrt{s}$, by relation (12). Thence, taking into consideration the determination (37), we find

$$|\mathbf{v}| = \frac{2\,u}{1+u^2}\,. \tag{39}$$

Then expressions (13) and (39) give

$$\mathbf{k}'^2 = \mu^2\,(u'_{\mathrm{rel}})^2\,, \tag{40}$$

where μ is the usual reduced mass, and

$$u'_{\rm rel} = \frac{2\,u}{\sqrt{1-u^2}} \tag{41}$$

is the relative velocity of an effective relativistic particle with mass m' emerging instead of the system of two particles. This result is found to be in full agrement with the physical meaning of Eq. (6), which is a relativistic generalization of the Lippmann-Schwinger equation in the spirit of Lobachevsky geometry. This equation describes the scattering of an effective relativistic particle on quasipotential $\widetilde{V}\,(\mathbf{p}',\,\mathbf{k}';\,E_{q'})$. The effective relativistic particle emerges instead of the system of two particles, has the mass m', the relative 3-momentum $\mathbf{k}'$ and carries the total c. m. energy of interacting particles, $\sqrt{s}$. Notice that the 3-momentum $\mathbf{k}'$ of an effective relativistic particle and hence its relative velocity (41), according to Eqs. (13) and (40), are an invariants of the Lorentz transformations.

Thus, in terms of relative velocity of an effective relativistic particle (41), the S-factor (32) and L-factor (36) are given by expressions

$$S_{\rm uneq}(u'_{\rm rel}) = \frac{X_{\rm uneq}(u'_{\rm rel})}{1-\exp\left[-X_{\rm uneq}(u'_{\rm rel})\right]}\,, \tag{42}$$

$$L_{\rm uneq}(u'_{\rm rel}) = \prod_{n=1}^{\ell}\left[1+\left(\frac{\alpha}{n\,u'_{\rm rel}}\right)^2\right]S_{\rm uneq}(u'_{\rm rel})\,, \quad X_{\rm uneq}(u'_{\rm rel}) = \frac{2\,\pi\,\alpha}{u'_{\rm rel}}\,. \tag{43}$$

The S-factor in Eq. (42) only formally has the same form, as the nonrelativistic S-factor (2). However, the S-factor in Eq. (42) has an obviously relativistic nature since as the argument r (the module of radius-vector $\mathbf{r}$) in the Coulomb potential (1) and the relativistic relative velocity of interacting particles, $\mathbf{v}$, (see [26]) both are relativistic invariants and hence the relative velocity of an effective relativistic particle (41), according to Eqs. (13) and (40), possesses this property as well.

4. Analysis relativistic threshold resummation S- and L-factors and summing up of the threshold singularities

The relativistic threshold resummation factors (32) and (36) [or (42) and (43)] has the following important properties:

• In the nonrelativistic limit, $u \ll 1$, they reproduces the well-known nonrelativistic result.

• In the relativistic limit, $u \to 1$, the factors (32) and (36) [or (42) and (43)] go to unity.

• In the case of equal masses they coincides with S-factor (3) and L-factor (4).

• The case when one of the particles is at rest means that $m_1 \to \infty$. This gives the following limited expression for the "velocity" u:

$$u \xrightarrow[m_1 \to \infty]{} \frac{|\mathbf{k}|}{\sqrt{{m_2}^2 + \mathbf{k}^2} + m_2}.$$

• In the ultrarelativistic limit, as it was argued in [30, 31], the bound state spectrum vanishes since mass of an effective relativistic particle $m' \to 0$. This feature reflects an essentialdifference between potential models and quantum field theory where an additional dimensional parameter appears Λ. One can conclude that within a potential model, the S- and L-factors which correspond to the continuous spectrum should go to unity in the limit $m' \to 0$. Thus, in contrast to the nonrelativistic case, the relativistic resummation factors, the S-factor (32) and L-factor (36) [or (42) and (43)] reproduces both the known nonrelativistic and the expected ultrarelativistic limits.

To illustrate the differences between the nonrelativistic S-factor (2) and the new relativistic S-factor in Eq. (42) in more detail, in Fig. 2 we plot the behavior of these factors as functions of u at different values of the parameter α (the numbers at the curves). The solid lines correspond to the S-factor in Eq. (42) and the dashed lines to the S-factor (2) with a substitution $v_{\rm nr} \to u$. From this figure one can see that in the region of nonrelativistic values of u, $u \leq 0.2$, where the influence of the S-factor is big, the difference between (42) and (2) is practically absent. However, when α increases, the nonrelativistic expression (2) gives a less suitable result in the region of large values u, in particular, as $u \to 1$. Thus,

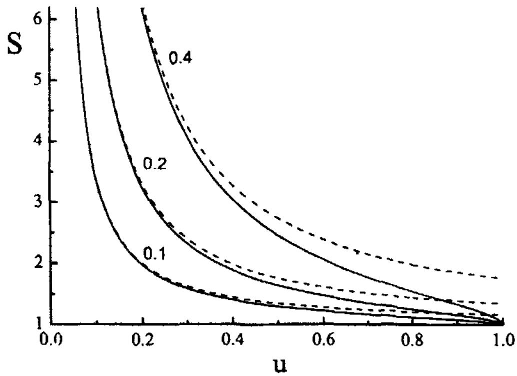

Fig. 2. Behavior of the S-factor at different values of the parameter α (the numbers at the curves). The solid lines correspond to the new relativistic S-factor (42) and the dashed lines to the nonrelativistic S-factor (2)

the above analysis demonstrates that the relativistic S-factor in Eq. (42), as would be expected, coincides in form with the nonrelativistic S-factor (2). However, the relative velocity of an effective relativistic particle (41) emerging instead of the system of two particles, now plays role of the parameter of velocity, but not the relativistic relative velocity of interacting particles, $\mathbf{v}$.

We note that these the new relativistic threshold resummation S- and L-factors could have a significant impact in interpreting strong-interaction physics. In many physically interesting cases the function $R(s)$, which is determined by the imaginary part of the quark current correlator, occurs as a factor in an integrand, as, for example, for the case of inclusive τ decay, for smearing quantities, and for the Adler D-function.

The principal contribution to the function $R(s)$ for the vector current with the S-factor can be written as

$$R(s) \to R_V^{(0)}(s) = \left[1 - \frac{(m_1 - m_2)^2}{s}\right]^2 \left[\frac{u\,(3 - u^2)}{2} + \frac{(m_1 - m_2)^2}{2s} u^3\right] S(u\,, \alpha), \tag{44}$$

where the total c. m. energy of interacting particles, $\sqrt{s}$, according to Eq. (37), can be expressed in terms of the "velocity" u as $s = [(m_1 + m_2)^2 - (m_1 - m_2)^2\, u^2]/(1 - u^2)$. The corresponding expression without the S-factor can be found in paper [32].

By using this formula, we study the influence of the S-factor to the function $R_V^{(0)}$. For Fig. 3 is shown dependence of the behaviour of the value $R_V^{(0)}$ with the new S-factor (42) as a function of dimensionless variable $\sqrt{s}/(m_1 + m_2)$ for different values of α (the numbers at the curves). Here a dashed line corresponds to the behaviour $R_V^{(0)}$ without S-factor that corresponds the case $\alpha = 0$ since $S(\alpha = 0) = 1$. This figure shows that the influence of the new S-factor (42) is much stronger in the threshold region and with growing energy $\sqrt{s}$ weakens, and all curves approach unity.

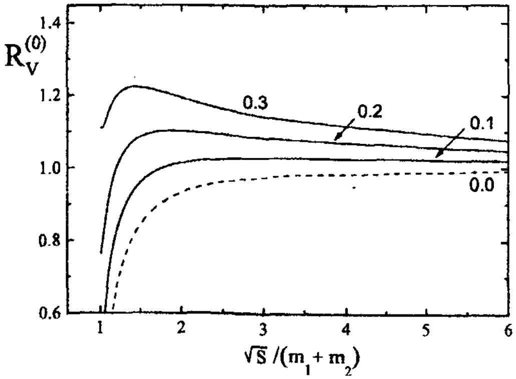

Fig. 3. Behavior of the function $R_V^{(0)}$ with the S-factor (42) as a function of dimensionless variable $\sqrt{s}/(m_1 + m_2)$ for different values of α (the numbers at the curves). The dashed line represents $R_V^{(0)}$ without the S-factor.

5. Conclusion

The new relativistic threshold resummation S- and L-factors (42) and (43) for the interaction of two relativistic particles of unequal masses were obtained. For this aim the relativistic quasipotential equation in relativistic configuration representation [25] with the Coulomb potential for the interaction of two relativistic particles of unequal masses was used.

The new relativistic threshold resummation factors obtained here reproduce both the known nonrelativistic and expected ultrarelativistic limits and correspond to the QCD-like Coulomb potential. The Coulomb potential only formally has the same form as the nonrelativistic potential but differs in the relativistic configuration representation since its behavior corresponds to the quark-antiquark potential $V_{q\bar{q}} \sim \bar{\alpha}_s(Q^2)/Q^2$ with the invariant charge $\bar{\alpha}_s(Q^2) \sim 1/\ln Q^2$. So, the principal effect coming from the running of the QCD coupling is accumulated.

The new S-factor coincides in form with the nonrelativistic S-factor (2); however, the role of the parameter of velocity is played not by the relative velocity of interacting particles, $\mathbf{v}$, but by the relative velocity (41) of an effective relativistic particle emerging instead of the system of two particles.

It was shown that there is a difference (see Fig. 2) between the expression (42) obtained here and the nonrelativistic S-factor (2). As the new relativistic resummation factors (42) and (43) were obtained within the framework of completely covariant method, one can expect that these factors takes into account more adequately relativistic nature of interaction.

We have suggested new expression for $R(s)$ in which threshold singularities are summarized by a potential contribution. It was demonstrated that the new relativistic resummation S-factor has the influence on the behavior of the function $R(s)$. Here the behaviour of S- and L-factors at intermediate values of u becomes important.

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