Письма в ЭЧАЯ. 2005. Т. 2, №4(127). С. 17–23

УДК 539.12.01

RELATIVISTIC RESUMMATION OF THRESHOLD SINGULARITIES IN THE QUASI-POTENTIAL APPROACH

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The quasi-potential approach in quantum field theory is used to derive relativistic threshold resummation factors in quantum chromodynamics. We also suggest a new model expression for $R(s)$ in which threshold singularities are summarized into a main potential contribution.

Квазипотенциальный подход в квантовой теории поля применяется для определения релятивистских факторов, суммирующих пороговые сингулярности в квантовой хромодинамике. Мы также предлагаем новое модельное выражение для $R(s)$, в котором пороговые сингулярности просуммированы в основной потенциальный вклад.

1. A description of quark–antiquark systems near threshold does not allow us to truncate the perturbative series even if the expansion parameter α_S is small. The problem is well known from QED [1]. The real expansion parameter in the threshold region is α/v. It becomes singular when the velocity $v \to 0$. These threshold singularities of the form $(\alpha/v)^n$ have to be summarized.

The resummation can be performed on the level of potential consideration. In the nonrelativistic approximation the Schrödinger equation with the Coulomb potential $V(r) = -\alpha/r$ leads to the known S-wave Gamov–Sommerfeld–Sakharov factor [2–4]

$$S_{\rm nr} = \frac{X_{\rm nr}}{1-\exp\left(-X_{\rm nr}\right)}, \qquad X_{\rm nr} = \frac{\pi\alpha}{v_{\rm nr}}, \tag{1}$$

which is related to the wave function of the continuous spectrum at the origin, $|\psi(0)|^2$. Here $v_{\rm nr}$ is the velocity of the particle. An expansion of (1) in a power series in the coupling constant α reproduces the threshold singularities of the form $(\alpha/v)^n$. The corresponding nonrelativistic expression can also be obtained for higher ℓ states (see, e.g., [5]).

In the relativistic theory the nonrelativistic approximation needs to be modified. For a systematic relativistic analysis of quark–antiquark systems, it is essential from the very beginning to have a relativistic generalization of the S factor. It is also important to take into account the difference between the Coulomb potential in the case of QED and the quark–antiquark potential in the case of QCD. The corresponding relativistic resummation of the S factor has been found in [6]. Its applications for describing some hadronic processes can be found in [7–9].

In this note we derive a relativistic P factor, propose and analyze expressions for $R(s)$ which include resummation of the threshold singularities.

2. The resummation factors appear in the parametrization of the imaginary part of corresponding quark current correlators, $R(s)$. In QED, the function $R(s)$ can be approximated by the Bethe–Salpeter (BS) amplitude of two charged particles, $\chi_{\rm BS}(x)$, at $x = 0$ [10]. The nonrelativistic replacement of this amplitude by the wave function, which obeys the Schrödinger equation with the Coulomb potential, leads to formula (1) with the substitution $\alpha \to 4\alpha_S/3$, for QCD.

A starting point of our consideration is the quasi-potential (QP) approach proposed by Logunov and Tavkhelidze [11], in the form suggested by Kadyshevsky [12]. To find an explicit form for the relativistic resummation factors, we use a transformation of the QP equation from momentum space into relativistic configuration space [13].

The possibility of using the QP approach for our task is based on the fact that the BS amplitude, which parameterizes the physical quantity $R(s)$, is taken at $x = 0$, therefore, in particular, at the relative time $\tau = 0$. The QP wave function is defined as the BS amplitude at $\tau = 0$, and R can be expressed in terms of the QP wave function $\psi_{\rm QP}(\mathbf{p})$ by using the relation

$$\chi_{\rm BS}(x=0) = \int d\Omega_p \psi_{\rm QP}(\mathbf{p}), \tag{2}$$

where $d\Omega_p = (d\mathbf{p})/[(2\pi)^3 E_p]$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid $E_p^2 - \mathbf{p}^2 = m^2$. In the following we will consider the case of two particles with the same masses m and use the system of units $c = \hbar = m = 1$.

The proper Lorentz transformation means a translation in the Lobachevsky space. The role of the plane waves corresponding to these translations are played by the following functions:

$$\xi(\mathbf{p}, \mathbf{r}) = (E_p - \mathbf{p}\cdot\mathbf{n})^{-1-ir}, \tag{3}$$

where $\mathbf{r} = \mathbf{n}r$ and $\mathbf{n}^2 = 1$. These functions correspond to the principal series of unitary representations of the Lorentz group and in the nonrelativistic limit ($p \ll 1$, $r \gg 1$) $\xi(\mathbf{p}, \mathbf{r}) \to \exp(i\mathbf{p}\cdot\mathbf{r})$.

The functions (3) obey certain conditions of completeness and orthogonality and serve to formulate the QP equation in the relativistic configuration r space. The local form of this equation is

$$\left(2E - 2\hat{H}_0\right)\psi(\mathbf{r}) = V(r)\psi(\mathbf{r}). \tag{4}$$

The free Hamiltonian is a finite difference operator [13, 14]

$$\hat{H}_0 = \cosh\left(i\frac{\partial}{\partial r}\right) + \frac{i}{r}\sinh\left(i\frac{\partial}{\partial r}\right) - \frac{\Delta_{\theta,\varphi}}{2r^2}\exp\left(i\frac{\partial}{\partial r}\right), \tag{5}$$

where $\Delta_{\theta,\varphi}$ is the angular part of the Laplacian operator.

Solutions of this equation, in principle, can contain arbitrary functions of r with period i, the so-called i-periodic constants, which appear in the solutions due to the finite difference nature of the Hamiltonian (5). For some problems, such as defining the bound state spectrum, this i-periodic constant is not important. However, for the purpose of extracting resummation factors, one must develop a method that avoids this ambiguity.

Consider the Coulomb potential defined in relativistic configuration space

$$V(r) = -\frac{\alpha}{r}. \tag{6}$$

The ξ transformation of (6) gives the potential in momentum space

$$V(\Delta) \sim \frac{1}{\chi_\Delta \sinh \chi_\Delta}, \tag{7}$$

where the relative rapidity χ_Δ corresponds to $\mathbf{\Delta} = \mathbf{p}(-)\mathbf{k}$ and is defined in terms of the square of the momentum transfer by $Q^2 = -(p-k)^2 = 2(\cosh \chi_\Delta - 1)$. For large Q^2 the potential $V(\Delta)$ behaves as $(Q^2 \ln Q^2)^{-1}$, which reproduces the principal behavior of the QCD potential proportional to $\bar{\alpha}_S(Q^2)/Q^2$ with $\bar{\alpha}_S(Q^2)$ being the QCD running coupling. This property of the quasi-potential (6), its QCD-like behavior, has been noted by Savrin and Skachkov in [18].

Solutions of Eq. (4) for the Coulomb potential have been investigated in [16]. Other forms of the QP equation with the Coulomb potential have been considered in [17]. To solve quasi-potential equation (4) with the potential (6), we use the method developed in [19] and [6]. This approach lead to the following relativistic S factor [6]:

$$S(\chi) = \frac{X(\chi)}{1 - \exp\left[-X(\chi)\right]}, \qquad X(\chi) = \frac{\pi\alpha}{\sinh \chi}, \tag{8}$$

where χ is the rapidity which related to s by $2\cosh\chi = \sqrt{s}$. The function $X(\chi)$ in Eq. (8) can be expressed in terms of the velocity $v = \sqrt{1 - 4/s}$, where $\sqrt{s}$ is the center-of-mass energy, as $X(\chi) = \pi\alpha\sqrt{1-v^2}/v$.

The S factor is involved to the expression for the function $R(s)$ that corresponds to the vector quark current. However, to perform a threshold resummation in the axial-vector case, one has to use the relativistic P factor, which corresponds to $\ell = 1$ states.

To derive this factor we note that in the nonrelativistic case the S factor is defined by the wave function at $r = 0$. In the relativistic case, one has to use the value of QP wave function at $r = i$ [6]. Indeed, according to Eqs. (2) and (3), one finds a relation between the required BS amplitude and the QP wave function, $\chi_{\mathrm{BS}}(x = 0) = \psi_{\mathrm{QP}}(r = i)$. Performing a partial-wave analysis, we further observe that the QP wave function for an ℓ state will contain the generalized power [14, 15]

$$(-r)^{(\ell+1)} = i^{l+1}\frac{\Gamma(ir + l + 1)}{\Gamma(ir)},$$

which vanishes at $r = i$ for $\ell \neq 0$. Thus, we need only to consider the $\ell = 0$ wave function for which we can write $\psi(\mathbf{r}) = \psi(r)$.

The P factor in the nonrelativistic case is defined by derivative of the wave function at $r = 0$. In the relativistic case, instead of the derivative, one has to use its finite difference analog

$$\Delta^* = \frac{1}{i}\left[\exp\left(i\frac{\partial}{\partial r}\right) - 1\right]. \tag{9}$$

Thus, the P factor is connected, as one can expect, with the $\ell = 1$ partial function. The corresponding QP wave function has the form

$$\varphi_1(r,\chi) = C_1(\chi)\frac{r}{r^{(2)}}\int_\alpha^\beta d\zeta\, \frac{\exp\left[(ir+2)\zeta\right]}{(\exp\zeta - \exp\chi)^4}\left[\frac{\exp\zeta - \exp(-\chi)}{\exp\zeta - \exp\chi}\right]^{-2+iA}, \tag{10}$$

where

$$A = \frac{\alpha}{2\sinh\chi}. \tag{11}$$

The contour of integration in the complex ζ plane in (10) and its end points α and β are the same as in [6]. The relativistic P factor is

$$P(\chi) = \lim_{r \to i} \left| \frac{3}{\sinh\chi} \, \Delta^* \left[\frac{\varphi_1(r,\chi)}{r} \right] \right|^2 . \tag{12}$$

The normalization constant $C_1(\chi)$ in (10) can be obtained as in [6].

By using Eqs. (9), (10) and (12), we finally find

$$P(\chi) = \left(1 + \frac{\alpha^2}{4\sinh^2\chi}\right) \frac{X(\chi)}{1 - \exp\left[-X(\chi)\right]}, \tag{13}$$

where $X(\chi)$ is defined in (8).

The relativistic threshold resummation factors (8) and (13) have the following important properties. In the nonrelativistic limit, $v \ll 1$, they reproduce the known nonrelativistic result. In the ultrarelativistic limit, as has been argued in [20,21], the bound state spectrum vanishes as a mass $m \to 0$ because the particle mass is the only dimensional parameter. This feature reflects an essential difference between potential models and quantum field theory, where an additional dimensional parameter appears. One can conclude that within a potential model, the S and P factors which correspond to the continuous spectrum should go to unity in the limit $m \to 0$. Thus, in contrast to the nonrelativistic case, the relativistic resummation factors, the S factor (8) and obtained here P factor (13), reproduce both the known nonrelativistic and the expected ultrarelativistic limits.

3. In comparing theoretical results with experimental data, it is important to use the «simplest» objects which allow one to check direct consequences of the theory without using model assumptions in an essential manner. In this case it is possible to justify transparently the validity of basic statements of the theory and make some conclusions about completeness and efficiency of the theoretical methods used. Some single-argument functions which have a straightforward connection with experimentally measured quantities can play the role of these objects. The corresponding functions can be defined with the Euclidean and the Minkowskian arguments [22].

The $R(s)$ function discussed above, which is determined by the imaginary part of the correlator of the vector or axial-vector quark current, $R_{V/A}(s)$, plays the role of this object. Corresponding perturbative expressions can be written as (see [23,24])

$$R_{V/A}^{PT}(s) = T_{V/A}(v) \left[1 + \frac{\alpha_S}{\pi} \, g_{V/A}(v)\right], \tag{14}$$

where

$$T_V(v) = v \, \frac{3 - v^2}{2}, \qquad g_V(v) = \frac{4\pi}{3} \left[\frac{\pi}{2v} - \frac{3+v}{4}\left(\frac{\pi}{2} - \frac{3}{4\pi}\right)\right], \tag{15}$$

$$T_A(v) = v^3, \qquad g_A(v) = \frac{4\pi}{3} \left[\frac{\pi}{2v} - \left(\frac{19}{10} - \frac{22}{5}v + \frac{7}{2}v^2\right)\left(\frac{\pi}{2} - \frac{3}{4\pi}\right)\right]. \tag{16}$$

The functions $g_V(v)$ [1] and $g_A(v)$ [25] approximate the corresponding exact two-loop expressions.

The perturbative representation (14) is not applicable in the threshold area because it does not contain the resummation of the threshold singularities. The expressions including the resummation factors have the form

$$\mathcal{R}_{V/A}(s) = \mathcal{R}^{(0)}_{V/A}(s) + \mathcal{R}^{(1)}_{V/A}(s) = \mathcal{R}^{(0)}_{V/A}(s)\left[1+\delta_{V/A}(s)\right], \tag{17}$$

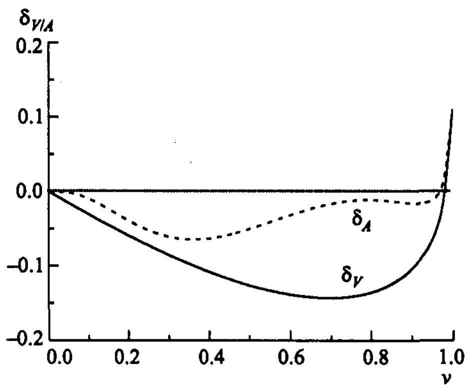
where

$$\mathcal{R}^{(0)}_{V}(s) = T_V(v)\,S(\chi), \qquad \mathcal{R}^{(0)}_{A}(s) = T_A(v)\,P(\chi), \tag{18}$$

$$\mathcal{R}^{(1)}_{V/A}(s) = T_{V/A}(v)\left[\frac{\alpha_S}{\pi}\,g_{V/A}(v) - \frac{1}{2}X(\chi)\right]. \tag{19}$$

In the limit $m \to 0$ the resummation factors $S, P \to 1$, the vector and axial-vector corrections become asymptotically equal, $\delta_{V/A}(s) \to \alpha_S/\pi$, and expression (17) reproduces the known massless formula.

The function $\mathcal{R}^{(0)}(s)$ in (17) is the product of the factor $T(v)$ and the threshold resummation factor for corresponding state. It describes the principal «potential» contribution. The next term $\mathcal{R}^{(1)}(s)$ in (17) relates to a QCD correction. The relative correction in (17) is described by $\delta_{V/A}(s)$. Its behavior is shown in the figure. The curves were obtained for $\alpha_S = 0.35$, which corresponds to the value of the strong coupling extracted from the τ-decay data (see [26]). The figure demonstrates that the correction to the principal potential contribution is small for a wide energy interval: $|\delta(s)| \lesssim 15\%$.



The relative corrections $\delta_{V/A}$ vs. v

4. To summarize the threshold singularities and find corresponding resummation factors, we have used the quasi-potential approach in quantum field theory. These relativistic resummation factors appear in the function $R(s)$, which is proportional to the imaginary part of the quark current correlator, and could have a significant impact in interpreting strong-interaction physics. Indeed, in many physically interesting cases, the function $R(s)$ occurs as a factor in an integrand, as, for example, for the case of inclusive τ decay, for smearing quantities, and for the Adler D function.

The relativistic threshold resummation factors obtained here reproduce both the known nonrelativistic and expected ultrarelativistic limits and correspond to the QCD-like Coulomb potential.

We have suggested new expressions for $R(s)$ in which threshold singularities are summarized by a potential contribution. We have demonstrated that the QCD correction to the principal potential contribution is rather small for a wide interval of energies.

Acknowledgements. The authors would like to thank Profs. A. N. Sissakian, D. V. Shirkov, and N. B. Skachkov for interest in this work and valuable discussions. Partial support of the work by the International Program of Cooperation between the Republic of Belarus and JINR, the State Program of Basic Research «Physics of Interactions» and the grant of the Ministry of Education is gratefully acknowledged. The work of I. L. S. was also supported in part by the Russian grants RFBR 02-01-00601 and NSh-2339.2003.2.

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Received on September 22, 2004.