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Threshold resummation S-factor for a system of two relativistic spinor particles with equal masses

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A new threshold resummation S-factor in quantum chromodynamics is obtained for a composite system of two relativistic spin-1/2 quarks having equal masses and interacting via a Coulomb-like chromodynamical potential. The pseudoscalar, vector, and pseudovector cases are considered. The present analysis is performed on the basis of the relativistic quasipotential approach in the Hamiltonian formulation of quantum field theory via a transition to the relativistic configuration representation for the case of a composite system formed by two relativistic spinor particles of equal masses.

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1. Introduction

By describing quark-antiquark systems near their production threshold $s = 4m^2$ we can not cut off the perturbative series even if the QCD coupling constant α_s is small [1]. The reason is that, in the threshold region, one has to construct a perturbation expansion in terms of α_s/v, which is a singular quantity, where

$$v = \sqrt{1 - \frac{4m^2}{s}} \tag{1}$$

is the relative velocity of fermions (or quarks) in the c.m. frame above threshold, $\sqrt{s}$ being the total c.m. energy of interacting particles and m there is their masses (we use

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the system of units where $\hbar = c = 1$). In order to obtain a meaningful result, the threshold singularities of the form $(\alpha_s/v)^n$ should be summed. In the nonrelativistic case for the Coulomb interaction

$$V(r) = -\frac{\alpha_s}{r} \tag{2}$$

these threshold singularities in the form $(\alpha_s/v)^n$ can be explicitly summarized by the known S-factor Gamov–Sommerfeld–Sakharov [2–4]

$$S_{\rm nr} = \frac{X_{\rm nr}}{1-\exp(-X_{\rm nr})}, \quad X_{\rm nr} = \frac{\pi\alpha_s}{v_{\rm nr}}; \tag{3}$$

which is related to the continuum wave function at the origin by $|\psi(0)|^2$ where $2v_{\rm nr}$ is the relative velocity of two nonrelativistic particles.

In relativistic theory, the nonrelativistic expression in (3) for two spinless particles of equal mass should be modified. For the first time the relativization of the S-factor (3) in QCD in the case of two particles of equal masses ($m_1 = m_2 = m$) was executed in [5, 6] and it consisted in the change $v_{\rm nr} \to v$. This factor was used for the description of effects close to the threshold of pair production in the processes $e^+e^- \to t\bar{t}$ and $e^+e^- \to W^+W^-$. Just same form of the S-factor but with the change $v_{\rm nr} \to v_{\rm rel} = 2v/(1+v^2)$ for the interaction of two particles of equal masses was later suggested in [7]. Another form of the relativistic generalization of the S-factor also in the case of two particles of equal masses was obtained in [8]. The relativistic S-factor for two particles of arbitrary masses ($m_1 \neq m_2$) was presented in [9]. This factor was derived within the framework of relativistic quantum mechanics on the basis of the Schrödinger equation.

The new method to the relativization of the S-factor in the case of two particles of equal masses was developed by Milton and Solovtsov in [10]. Their the method is based on the relativistic quasipotential (RQP) approach proposed by Logunov and Tavkhelidze [11] in the form suggested by Kadyshevsky [12]. This approach is a new step to application of the quasipotential approach in QCD and it gives the following expression for the relativistic S-factor:

$$S(\chi) = \frac{X(\chi)}{1-\exp\left[-X(\chi)\right]}, \quad X(\chi) = \frac{\pi\alpha_s}{\sinh\chi}, \tag{4}$$

where χ is the rapidity related to the total c.m. energy of interacting particles, $\sqrt{s}$, by

$$\sqrt{s} = 2m\cosh\chi. \tag{5}$$

The function $X(\chi)$ in Eq. (4) can be expressed in terms of v as $X(\chi) = \pi\alpha_s\sqrt{1-v^2}/v$. However, it was shown in [13, 14] that the relativistic limits of the S-factors in [7–9] differ substantially from the relativistic limit ($v \to 1$) of the S-factor in (4) which tends to unity in the relativistic limit.

We note that in the method developed in [10], the possibility of transformation of quasipotential (QP) equation from momentum space into relativistic configurational representation in the case of two particles of equal masses [15] has been used also. Moreover, it is important the potential (2) that used by them possesses the QCD-like behaviour [16].

Applications of the factor (4) for describing some hadronic processes can be found in [17–19]. Recently, the relativistic S-factor (4) has been applied in [20] to reanalyze the mass limits obtained for magnetic monopoles which might have been produced at the Fermilab Tevatron.

The resummation factors appears too in the parametrization of the imaginary part of the corresponding quark current correlators, in the Drell ratio $R(s)$ [21], which in the two-particle approximation can be approximated in terms of the Bethe–Salpeter (BS) amplitude of two charged particles $\chi_{\rm BS}(x)$ at $x = 0$ [22].

The nonrelativistic replacement of this amplitude by the wave function, which obeys the Schrödinger equation with the Coulomb potential (2), leads to formula (3) with a substitution $\alpha_s \to 4\alpha_s/3$ for QCD. The possibility of using the RQP approach for our task is based on the fact that the BS amplitude, which parameterizes the physical quantity $R(s)$ and is taken at $x = 0$ and hence at the relative time $\tau = 0$, can be expressed in the case of the interaction of two relativistic particles of equal masses m through the wave RQP function in the momentum space, $\Psi_q(\mathbf{p})$, and in the configuration representation, $\psi_q(\mathbf{r})$, as

$$\chi_{\rm BS}(x=0) = \frac{1}{(2\pi)^3}\int d\Omega_{\mathbf{p}}\Psi_q(\mathbf{p}) = \psi_q(\mathbf{r})\Big|_{r=i\lambda}, \tag{6}$$

where $\lambda = 1/m$ is the Compton wavelengs particle of mass m, $d\Omega_{\mathbf{p}} = (md\mathbf{p})/p_0$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid $p_0^2 - \mathbf{p}^2 = m^2$.

The purpose of this paper is to generalize the method proposed in [10, 13, 14] for the case a composite system formed by two relativistic quarks having equalmasses and a spin of 1/2 and interacting via a Coulomb-like chromodynamic potential (2). The pseudoscalar, vector, and pseudovector cases are considered, and the behavior of the S-factor is analyzed in the nonrelativistic, relativistic, and ultrarelativistic cases. This investigation be based on the RQP approach to quantum field theory [11] in the form proposed in [12] via a transition from the momentum representation in Lobachevsky space to the three-dimensional relativistic configuration representation introduced in [15] for a composite system of two relativistic equal-mass particles.

2. Coulomb wave function: the case of two relativistic spinor particles of equal masses

The present analysis be based on the fully covariant two-particle three-dimensional RQP equation in an integral form for the case of two relativistic spinor equal-mass particles. In the configuration representation, this equation for the radial RQP wave function of relative orbital angular momentum $\ell = 0$ has the form [23]

$$\int\limits_0^\infty d\chi'\,(\cosh\chi - \cosh\chi')\sin\rho\chi' \int\limits_0^\infty d\rho' \sin(\rho'\chi')\varphi_0(\rho',\chi) = \tag{7}$$

$$= V(\rho)\int\limits_0^\infty d\chi' \hat{A}(\cosh\chi')\sin\rho\chi' \int\limits_0^\infty d\rho' \sin(\rho'\chi')\varphi_0(\rho',\chi),$$

where the rapidity χ is introduced, as before, via relation (5), the potential $V(\rho)$, where $\rho = r/\lambda$, is local in the sense of Lobachevsky's geometry, and the operator $\hat{A}$ is given by

$$\hat{A}\left(\frac{\widehat{\Pi}_0}{2m}\right) = \frac{1}{4}\left[a\left(\frac{\widehat{\Pi}_0}{2m}\right)^2 + b\right], \tag{8}$$

$$a = \begin{cases} 1 & \text{for } \hat{O} = \gamma_5, \text{(pseudoscalar)}; \\ \frac{1}{2} & \text{for } \hat{O} = \gamma_\mu, \text{(vector)}; \\ -\frac{1}{2} & \text{for } \hat{O} = \gamma_5\gamma_\mu, \text{(pseudovector)}; \end{cases} \qquad b = \begin{cases} 0 & \text{for } \hat{O} = \gamma_5, \text{(pseudoscalar)}; \\ \frac{1}{4} & \text{for } \hat{O} = \gamma_\mu, \text{(vector)}; \\ \frac{3}{4} & \text{for } \hat{O} = \gamma_5\gamma_\mu, \text{(pseudovector)}. \end{cases}$$

A solution of RQP equation (7) with the potential (2) is sought in the form

$$\varphi_0(\rho, \chi) = \int_{\alpha_-}^{\alpha_+} d\zeta e^{i\rho\zeta} R_0(\zeta, \chi), \tag{9}$$

where integration is performed in a complex plane along a contour between the end points $\alpha_\pm = -R \pm i\varepsilon, R \to +\infty, \varepsilon \to +0$, the points $\pm\chi + 2\pi ni\,(n = 0, \pm 1, \ldots)$ are the branch points of the function $R_0(\zeta, \chi)$, and the contour of integration must not intersect cuts which we take from $-\infty + 2\pi ni$ to $\pm\chi + 2\pi ni$. (see Fig. 1), that is, in just the same way as in [10, 13, 14, 23, 24]. Substituting Eqs. (2) and (9) into Eq. (7), taking into account

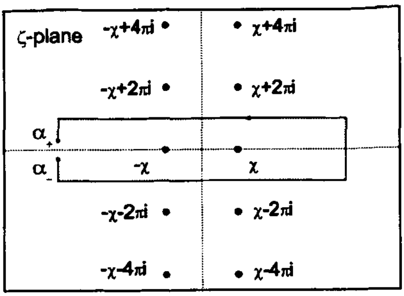

Figure 1: Contour of integration in Eq. (9) and singularities of the function (12) in the complex ζ-plane.

that

$$\frac{1}{i\pi}\int_0^\infty d\rho' \sin(\rho'\chi')e^{i\rho'\zeta} = \frac{1}{i\pi}\frac{\chi'}{\chi'^2 - \zeta^2},$$

and performing integration by parts, we arrive to the equation

$$\frac{d}{d\zeta}\left[(\cosh\chi - \cosh\zeta)\, R_0(\zeta, \chi)\right] - i\tilde{\alpha}_s\hat{A}(\cosh\zeta)R_0(\zeta, \chi) = 0, \quad \tilde{\alpha}_s = m\alpha_s, \tag{10}$$

with the boundary condition

$$e^{i\rho\zeta}\left(\cosh\chi-\cosh\zeta\right)R_0(\zeta,\chi)\Big|_{\zeta=\alpha_-}^{\zeta=\alpha_+}=0. \tag{11}$$

As a result the solution of Eq. (10) with the boundary condition (11) is

$$R_0(\zeta,\chi)=C_0(\chi)\frac{\exp\left[-\frac{i\tilde{\alpha}_s a}{4}\sinh\zeta+(1-i\tilde{\rho})\zeta+iB\chi\right]}{(e^{\zeta}-e^{\chi})^2}\left[\frac{e^{\zeta}-e^{-\chi}}{e^{\zeta}-e^{\chi}}\right]^{-1+iB}, \tag{12}$$

where $C_0(\chi)$ is an arbitrary function of χ, the parameters a, b and $\tilde{\alpha}_s$ are defined in Eqs. (8) and (10), and the parameters $\tilde{\rho}$ and B are given by

$$\tilde{\rho}=\frac{\tilde{\alpha}_s a\cosh\chi}{4},\quad B=\frac{\tilde{\alpha}_s(a\cosh^2\chi+b)}{4\sinh\chi}. \tag{13}$$

At $\chi=i\kappa$, the parameter B is related to the quantization condition [23]

$$\frac{\tilde{\alpha}_s(a\cos^2\kappa+b)}{4\sin\kappa}=n,\ n=1,2,\ldots,0<\kappa<\pi/2. \tag{14}$$

In the case when the interaction vanishes, $\alpha_s\to 0$, the solution $\varphi_0(\rho,\chi)$ should reproduce the known free wave function

$$\lim_{\alpha_s\to 0}\varphi(\rho,\chi)\xrightarrow[\rho\to\infty]{}\frac{\sin(\rho\chi)}{\sinh\chi}. \tag{15}$$

Substituting the solution (12) into (9) and performing ζ-integration in the complex plane along a contour with end points $\alpha_\pm$ (in the same way as in [10, 13, 14, 23, 24]) we obtain the resulting solution which does not contain the i-periodic constant:

$$\varphi_0(\rho,\chi)=2\,C_0(\chi)\,e^{iB\chi}\sinh\left[\pi(\rho-\tilde{\rho})\right]\times \tag{16}$$
$$\times\int_{-\infty}^{\infty}dx\frac{\exp\left[\frac{i\tilde{\alpha}_s a}{4}\sinh x+\left(1+i(\rho-\tilde{\rho})\right)x\right]}{(e^{x}+e^{\chi})^2}\left[\frac{e^{x}+e^{-\chi}}{e^{x}+e^{\chi}}\right]^{-1+iB}.$$

One can easily find that the normalization factor $C_0(\chi)$ in the solution given by (16) is real-valued. In addition, we see that, apart from the oscillating factor $\exp[i\tilde{\alpha}_s a\sinh x/4]$, the solution in (16) coincides in form with the solution in the spinless case, and at $a=0$ and $b=2$, the former reduces to the latter (see [13, 14]). Taking the foregoing into consideration, we will set the oscillating factor $\exp[i\tilde{\alpha}_s a\sinh x/4]$ to unity in the exact solution in (16). Such aproximation not only does not break characteristic to symmetries of the solution in (16), but also allows to present the expression for the RQP radial wave function for the s-wave state in terms of a hypergeometric function as (in the same way as in [13, 14] for the spinless case)

$$\varphi_0(\rho,\chi)=2\pi C_0(\chi)e^{iB\chi-\chi+i(\rho-\tilde{\rho})\chi}(\rho-\tilde{\rho})F\big(1-iB,1-i(\rho-\tilde{\rho});2;1-e^{-2\chi}\big), \tag{17}$$

where the parameters $\tilde{\rho}$ and B were defined in Eqs. (13), and the real-valued normalization factor $2\pi C_0(\chi)$ gives by the expression

$$|2\pi C_0(\chi)|^2 = e^{\pi B}|\Gamma(1 - iB)|^2, \tag{18}$$

which one can derive from the boundary condition in (15) and the asymptotic expression

$$\varphi_0(\rho,\chi)|_{\rho\gg 1} \approx \frac{2\pi C_0(\chi)e^{-\pi B/2}}{\sinh\chi|\Gamma(1-iB)|} \sin\left\{(\rho-\tilde{\rho})\chi + B\ln\left[2(\rho-\tilde{\rho})\sinh\chi\right] + \arg\Gamma(1-iB)\right\}.$$

3. Threshold S-factor for a system of two relativistic spinor particles of equal masses

We define the threshold S-factor in the spinor case as

$$S_{\mathrm{RQP,S}}(\chi) = \lim_{\rho\to i}\left|e^{-\pi\tilde{\rho}/2}\Gamma(1+i\tilde{\rho})\frac{\varphi_0(\rho,\chi)}{\rho}\right|^2, \tag{19}$$

where not only does the additional factor $\exp(-\pi\tilde{\rho}/2)\Gamma(1+i\tilde{\rho})$ lead to the correct relativistic limit for $\chi\to+\infty$, which is equal to unity, but it also ensures a transition to the spinless case at $a=0$ and $b=2$. Thus, we see that, in the spinor case, the function

$$\psi_0(\rho,\chi) = e^{-\pi\tilde{\rho}/2}\Gamma(1+i\tilde{\rho})\varphi_0(\rho,\chi)$$

is a physical wave function for the Coulomb interaction (2).

Since the BS amplitude $\chi_{\mathrm{BS}}(x=0)$ is related to the RQP wave function $\psi_q(\mathbf{r})$ by Eq. (6), the following expression (see [25]) for the relativistic threshold S-factor in the case of a composite system formed by two relativistic spinor particles of equal masses can be obtained with the aid of relations (17)–(19)

$$S_{\mathrm{RQP,S}}(\chi) = \frac{X_{\mathrm{RQP,S}}(\chi)}{1-\exp\left[-X_{\mathrm{RQP,S}}(\chi)\right]}e^{-\pi\tilde{\rho}}\left|\Gamma(2+i\tilde{\rho})F(1+iB,-i\tilde{\rho};2;1-e^{-2\chi})\right|^2. \tag{20}$$

Here, the quantity

$$X_{\mathrm{RQP,S}}(\chi) = 2\pi B = \frac{\pi\tilde{\alpha}_s(a\cosh^2\chi+b)}{2\sinh\chi} \tag{21}$$

can be expressed in terms of the velocity in (1) as

$$X_{\mathrm{RQP,S}}(v) = \frac{\pi\tilde{\alpha}_s(a+b-bv^2)}{2v\sqrt{1-v^2}}. \tag{22}$$

It is noteworthy that, at $a=0$ and $b=2$, the relativistic threshold resummation S-factor (20) goes over to the spinless S-factor (4), which, in the nonrelativistic limit ($v\ll 1$), reproduces the well-known nonrelativistic result (3) (see [13, 14]).

Let us study the behavior of the relativistic threshold resummation S-factor (20) in the nonrelativistic ($\chi\to+0$), relativistic ($\chi\to+\infty$), and ultrarelativistic limits. In the nonrelativistic limit, we have the expression

$$S_{\mathrm{RQP,S}}(\chi)|_{\chi\to+0} \approx \frac{\pi\tilde{\alpha}_s(a+b)/2\sinh\chi}{1-\exp[-\pi\tilde{\alpha}_s(a+b)/2\sinh\chi]}\frac{\pi\tilde{\alpha}_s a/2}{\exp(\pi\tilde{\alpha}_s a/2)-1}\left(1+\frac{\tilde{\alpha}_s^2a^2}{16}\right),$$

which, as was mentioned above, reduces to the spinless S-factor (4) at $a = 0$ and $b = 2$.

In the relativistic limit ($v \to 1$), we have

$$S_{\mathrm{RQP,S}}(\chi)|_{\chi\to+\infty} \approx \frac{2\pi(B-\tilde{\rho})}{1-\exp[-2\pi(B-\tilde{\rho})]} \xrightarrow[\chi\to+\infty]{} 1+0.$$

This expression is valid at all values of the spin parameters a and b in (8).

As was proven in [26, 27], the spectrum of bound states vanishes in the ultrarelativistic limit for $m \to 0$, since the particle mass is the only dimensional parameter. This feature reflects a substantial difference between potential models and quantum field theory, where there arises an additional dimensional parameter. In addition, we can conclude that the S-factor corresponding to the continuous spectrum should tend to unity as $m \to 0$.

Thus, we have established the dependence of the relativistic threshold resummation S-factor (20) on the spin parameters a and b. The above analysis of its behavior in the nonrelativistic ($v \ll 1$), relativistic ($v \to 1$), and ultrarelativistic ($m \to 0$) cases has shown that this resummation factor reproduces both the well-known nonrelativistic limit in the spinless case, where $a = 0$ and $b = 2$, and the expected relativistic and ultrarelativistic limits for all of the three cases: the pseudoscalar, vector, and pseudovector ones.

4. Conclusion

In the present study, a new threshold resummation S-factor, that in (20), has been obtained for a composite system of two relativistic spinor particles (quarks) having equal masses and interacting via a Coulomb-like chromodynamic potential. The pseudoscalar, vector, and pseudovector cases have been considered. For this purpose, a fully covariant Hamiltonian formulation of the quasipotential approach in quantum field theory has been implemented via a transition to the three-dimensional relativistic configuration representation [15] for the case of a composite system formed by two relativistic spinor particles of equal masses. The dependence of the relativistic threshold resummation S-factor (20) on the spin parameters a and b has been found. It has been shown that, at $a = 0$ and $b = 2$, the relativistic S-factor (20) reduces to the spinless S-factor in (4).

The present analysis of the behavior of the S-factor (20) in the nonrelativistic ($v \ll 1$), relativistic ($v \to 1$), and ultrarelativistic ($m \to 0$) limits has shown that it reproduces both the known nonrelativistic limit in the spinless case of $a = 0$ and $b = 2$ and the expected relativistic and ultrarelativistic limits for all of three cases: the pseudoscalar, vector, and pseudovector ones.

Since the expression (20) has been obtained here for the S-factor within a fully covariant method, it can be expected that the relativistic character of interacting particles and spin effects have been comprehensively taken into account. However, for completeness, it is necessary to consider the case of unequal masses of relativistic particles, this we plan to do.

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