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Leptonic Decay Widths for the Composite System of Two Spin Particles with Arbitrary Masses in the Relativistic Quasipotential Approach

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The new relativistic WKB expression for leptonic decay widths of the vector mesons as the relativistic component system of two fermions of arbitrary masses interacting by means of the funnel-type potentials is obtained. Quasipotential equation is solved by the WKB approximation. The behavior of the relativistic leptonic decay widths of vector mesons was investigated. Comparison of the behavior for new expression with its relativistic spinless analogue is given. Consideration is conducted within the framework of completely covariant quasipotential approach in the Hamiltonian formulation of quantum field theory, via a transition to the relativistic configurational representation in the case of two relativistic spin particles of arbitrary masses.

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1. Introduction

The present study is a continuation of studies performed earlier and reported in [1–3], where were obtained a relativistic expressions for the leptonic decay widths of vector mesons in the s-wave state ($\ell = 0$) to a lepton–antilepton pair on the basis of relativistic analog of semiclassical (WKB) method (see, for example, [4, 5]). In this studies was used the Logunov–Tavkhelidze relativistic quasipotential (RQP) approach [6] to the problem of a composite system of two relativistic spin particles to quantum field theory in the form proposed by V.G. Kadyshevsky [7] which is based on the completely covariant Hamiltonian formulation of quantum field theory via a transition to the three-dimensional relativistic configuration representation ($\mathbf{r}$ representation) [8–10]. In RQP approach [7] the relativistic WKB expression for the leptonic decay widths of vector meson V with energy $M_Q = M_n$ for given level n and with relative orbital moment $\ell \geq 0$ for the case of two relativistic spinless particles with arbitrary masses m_1, m_2, that interacts by means of potential

$$V(r) = V_{\rm conf}(r) - \frac{\alpha_s}{r}, \tag{1}$$

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where $V_{\text{conf}}(r)$ is the confining potential ($V_{\text{conf}}(0) = 0$), and α_s is the strong coupling constant, has following the form [1]

$$\Gamma_{n,\ell}(V \to e^+e^-) = \frac{2\alpha^2 Q_V^2 (\hbar c)^3 \Gamma^2(\ell+1)}{\pi \lambda'^3 g' m' c^2 \Gamma^2(2\ell+2) M_n^2} \left(\frac{u'_{\text{rel},n}}{g'}\right)^{2\ell+1} L_{\text{RQP}}\left(u'_{\text{rel},n}\right) \frac{dM_n}{dn}. \tag{2}$$

Here α is the fine-structure constant, Q_V is multiplier conditioned by isotopic structure of vector meson V and charges of quarks expressed in unit of the electric charge e and for the spinless quarks $Q_V = e_q$, where e_q is the quark charge in the units the electric charge e with the number of its colours $N_c = 3$; $\lambda' = \hbar/m'c$ is the Compton wavelengths of the effective relativistic particle with mass $m' = \sqrt{m_1 m_2}$, a relative three-momentum $\mathbf{q}'$ and the energy $E_{q'} = c\sqrt{m'^2c^2 + \mathbf{q}'^2}$, emerging instead of the system of two particles and carrying the total c.i.s. energy $\sqrt{s} = M_Q = \sqrt{Q^2} = \sqrt{(q_1+q_2)^2}$ of the interacting particles proportional to the energy $E_{q'}$ as $\sqrt{s} = c\sqrt{m_1^2c^2 + \mathbf{q}^2} + c\sqrt{m_2^2c^2 + \mathbf{q}^2} = 2g'E_{q'}$ (see [9, 10]), and the factor g' gives by formula

$$g' = \frac{m'}{2\mu} = \frac{m_1 + m_2}{2\sqrt{m_1 m_2}} \tag{3}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass,

$$u'_{\text{rel}} = \frac{2u}{\sqrt{1-u^2}} \tag{4}$$

is the relative velocity of the effective relativistic particle of mass m', where the velocity u determine by expression

$$u = \sqrt{1 - \frac{4m'^2c^4}{M_Q^2 - (m_1 - m_2)^2c^4}}, \tag{5}$$

and the relativistic spinless resummations Coulomb L and S factors as functions of the rapidity χ', which appears in considered RQP approach, give by expressions [11–13]1)

$$L_{\text{RQP}}(\chi') = \prod_{n=1}^{\ell} \left[1 + \left(\frac{\tilde{\alpha}_s}{2n\sinh\chi'}\right)^2\right] S_{\text{RQP}}(\chi'), \tag{6}$$

$$S_{\text{RQP}}(\chi') = \frac{X_{\text{RQP}}(\chi')}{1 - \exp[-X_{\text{RQP}}(\chi')]}, \quad X_{\text{RQP}}(\chi') = \frac{\pi\tilde{\alpha}_s}{\sinh\chi'}. \tag{7}$$

Here $\tilde{\alpha}_s = \alpha_s/g'\hbar c$, the value $\sinh\chi'$ is expressed through relative velocity in (4) as

$$\sinh\chi' = u'_{\text{rel}}/2g', \tag{8}$$

and χ' is the rapidity, which is connected to the 3-momentum $\mathbf{q}'$, the energy $E_{q'}$ of the effective relativistic particle of mass m' and the total c.i.s. energy M_Q of the interacting particles as

$$\mathbf{q}' = m'c\sinh\chi'\mathbf{n}_{q'}, \; |\mathbf{n}_{q'}| = 1, \; E_{q'} = m'c^2\cosh\chi', \; M_Q = 2g'm'c^2\cosh\chi'.$$

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1) We shall remind that the relativistic spinless resummations Coulomb L and S factors in (6) and (7) have the correct relativistic ($u \to 1$) and ultrarelativistic ($m' \to 0$) limits equal to unit unlike the relativistic S factors, presented in Refs. [14–16] (for details, see Refs. [11–13]).

In this paper within the framework of completely covariant of the quasipotential RQP approach in Hamiltonian formulation of quantum field theory for the case of two relativistic spin particles with arbitrary masses m_1, m_2 [7–10], we obtain the expression for the leptonic decay widths of vector mesons with relative orbital moment $\ell = 0$. For this purpose, the fully covariant finite-difference RQP equation in **r**-representation [9, 10] for the case of two relativistic spin particles of arbitrary masses m_1, m_2, that interacts by means of potential (1) is solved by the relativistic WKB method [4]. The condition of applicability of the relativistic WKB method is established. In Section 3, an expression for the leptonic decay widths of vector mesons in the s-wave state to a lepton–antilepton pair is obtained. In Section 4, the behavior of the new expression for the leptonic decay widths of vector mesons is compared with its relativistic spinless analog. The influence of spin parameters vector mesons in the s-wave state on the behavior of their the relativistic leptonic decay widths was investigated. The calculation of leptonic decay widths for the ground level of s-wave state of vector ρ_0, ω and ϕ-mesons that are formed by the corresponding pair of relativistic u-, d- and s-quarks, interacting by means of sum linear and Coulomb-like chromodynamical potentials was executed. The results of our investigations are discussed in the Conclusions.

2. The solutions of RQP equation in WKB approximation

The basis of our consideration is completely covariant RQP equation into the **r** representation in terms finite differences constructed in [17] for the radial RQP wave function $\varphi_\ell(r,\chi')$ that describes the composite system of two relativistic spin particles of arbitrary masses m_1, m_2 with the relative orbital moment $\ell \geq 0$. For spherically symmetric potential this equation has the form

$$\left(\widehat{H}^{\mathrm{rad}}_{0,\ell} - \cosh\chi'\right)\varphi_\ell(r,\chi') = -V(r)\hat{A}\left(\widehat{H}^{\mathrm{rad}}_{0,\ell}\right)\varphi_\ell(r,\chi'). \tag{9}$$

Here

$$\widehat{H}^{\mathrm{rad}}_{0,\ell} = \cosh\left(i\lambda'\frac{d}{dr}\right) + \frac{\lambda'^2\ell(\ell+1)}{2r(r+i\lambda')}\exp\left(i\lambda'\frac{d}{dr}\right)$$

is the radial part of operator free Hamiltonian

$$\widehat{H}_0 = 2m'c^2\left[\cosh\left(i\lambda'\frac{\partial}{\partial r}\right) + \frac{i\lambda'}{r}\sinh\left(i\lambda'\frac{\partial}{\partial r}\right) - \frac{\lambda'^2}{2r^2}\Delta_{\theta,\varphi}\exp\left(i\lambda'\frac{\partial}{\partial r}\right)\right],$$

while $\Delta_{\theta,\varphi}$ is its the angular part, potential $V(r)$ is local in the sense of Lobachevsky geometry and for simplicity depends not from energy M_Q, the operator $\hat{A}$ is defined by expression

$$\hat{A}\left(\widehat{H}^{\mathrm{rad}}_{0,\ell}\right) = \frac{1}{4}\left[a'\left(\widehat{H}^{\mathrm{rad}}_{0,\ell}\right)^2 + b'\right],$$

in which the spin parameters a' and b' for vector mesons has the following importances

$$a' = \frac{1}{2}g'^2,\ b' = \frac{3}{4} - \frac{1}{2}g'^2\ \mathrm{for}\ \hat{O} = \gamma_\mu\ (\mathrm{vector}), \tag{10}$$

and χ' is the rapidity, which is connected to the 3-momentum $\mathbf{\Delta}_{q',m'\lambda_Q}$ and the energy M_Q as[2]

$$\mathbf{\Delta}_{q',m'\lambda_Q} = m'c \sinh\chi' \mathbf{n}_{\Delta_{q',m'\lambda_Q}},\ |\mathbf{n}_{\Delta_{q',m'\lambda_Q}}| = 1,\ M_Q = 2g'\Delta^0_{q',m'\lambda_Q},\ \Delta^0_{q',m'\lambda_Q} = m'c^2\cosh\chi'. \tag{11}$$

We note that the values of the parameters a', b' in the Eq. (10) at $m_1 = m_2 = m$ coincides with the values of their the analogs a, b that were obtained for the case of spin and equal masses in [18].

We will seek the WKB solution of RQP Eq. (9) in the usual form [1–5]

$$\varphi_\ell(r,\chi') = \exp\left[\frac{i}{\hbar}g(r)\right],\ g(r) = g_0(r) + \frac{\hbar}{i}g_1(r) + \left(\frac{\hbar}{i}\right)^2 g_2(r) + \cdots. \tag{12}$$

With two first terms of the expansion in (12) the WKB solutions with the left, r_L, and the right, r_R, of the classical turning points in the inner region $r_L \le r \le r_R$ has then the form

$$\varphi_\ell^{L,R}(r,\chi') = \frac{C_{L,R}(\chi')}{2\sqrt[4]{[\mathcal{X}^2(r) - R^2(r)][1 + a'V(r)X(r)]}}\left\{\exp\left[i\alpha_+^{L,R}(r) \mp \frac{i\pi}{4}\right] + \right. \tag{13}$$
$$\left. + \exp\left[i\alpha_-^{L,R}(r) \pm \frac{i\pi}{4}\right]\right\},$$

where

$$\alpha_\pm^{L,R}(r) = \frac{1}{\chi'}\int\limits_{r_{L,R}}^{r} dr'\chi_\pm(r'),\ \chi_\pm(r) = \ln\left[\mathcal{X}(r) \pm \sqrt{\mathcal{X}^2(r) - R^2(r)}\right], \tag{14}$$

$$\mathcal{X}(r) = \frac{2X(r)}{1 + \sqrt{1 + a'V(r)X(r)}},\ X(r) = \cosh\chi' - \frac{b'}{4}V(r),\ R(r) = \sqrt{1 + \frac{\chi'^2\Lambda^2}{r^2}},\ \Lambda = \ell + 1/2,$$

$C_{L,R}$ are the normalization constants, and the turning points, $r_{L,R}$, are branch points of root in the WKB solutions (13), (14) that lead to the condition

$$\mathcal{X}(r_{L,R}) = R(r_{L,R}). \tag{15}$$

Condition of applicability of the relativistic WKB solutions (13) has the form

$$\chi'\left|\frac{\cosh\chi_{\text{eff}}(r)}{\chi_+(r)\sinh\chi_{\text{eff}}(r)}\frac{d\chi_+(r)}{dr}\right| \ll 1, \tag{16}$$

where

$$\chi_{\text{eff}}(r) = \text{arcosh}\mathcal{X}_{\text{eff}}(r) = \ln\left(\mathcal{X}_{\text{eff}}(r) + \sqrt{\mathcal{X}^2_{\text{eff}}(r) - 1}\right),\ \mathcal{X}_{\text{eff}}(r) = \cosh\chi_{\text{eff}}(r) = \frac{\mathcal{X}(r)}{R(r)}.$$

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[2] We shall remind that here all the 4-momentums belong to the upper sheet of the mass hyperboloid $\Delta^2_{q',m'\lambda_Q} = \Delta^{02}_{q',m'\lambda_Q} - c^2\mathbf{\Delta}^2_{q',m'\lambda_Q} = m'^2c^4$, where $\lambda_Q = (\lambda^0_Q; \boldsymbol{\lambda}_Q) = Q/\sqrt{Q^2}$ is the 4-velocity of a composite particle with the 4-momentum $Q = q_1 + q_2$, and $\Delta^0_{q',m'\lambda_Q}, \mathbf{\Delta}_{q',m'\lambda_Q}$ are, respectively, the time and spatial components of the 4-vector $\Lambda^{-1}_{\lambda_Q}q' = \Delta_{q',m'\lambda_Q}$ from the Lobachevsky space (for details, see Ref. [17]).

In the case of $\ell = 0$, the condition (16) is converted in inequality

$$\lambda' \left| \frac{\cosh \chi_S(r)}{\chi_S(r) \sinh \chi_S(r)} \frac{d\chi_S(r)}{dr} \right| \ll 1, \tag{17}$$

where

$$\chi_S(r) = \text{arcosh}\mathcal{X}(\text{r}) = \ln \left[\mathcal{X}(\text{r}) + \sqrt{\mathcal{X}^2(\text{r}) - 1} \right] \tag{18}$$

is the rapidity of relativistic particle of the mass m' that moves in the field of potential $V(r)$.

In conclusion this section we bring the WKB quantization condition of levels energy [19]

$$\int\limits_{r_L}^{r_R} dr \left[\chi_+(r) - \ln R(r) \right] = \pi\lambda' \left(n + \frac{1}{2} \right), \; n = 0, 1, \ldots, \; \ell \geq 0, \tag{19}$$

which, either as in the spinless case [1], is found from condition of the coincidence wave functions in Eq. (13) in point $r \in (r_L; r_R)$. For this we chooses

$$C_L = C_\ell \exp \left[-\frac{i}{\lambda'} \int\limits_{r_L}^{r} dr' \ln R(r') \right], \; C_R = C_\ell (-1)^n \exp \left[-\frac{i}{\lambda'} \int\limits_{r_R}^{r} dr' \ln R(r') \right],$$

where C_ℓ is the normalization constant. At $a' = 0, b' = 2/g'm'c^2$ the WKB quantization condition of levels energy in Eq. (19) coincides with the analogous expression obtained in the case of spinless particles of arbitrary masses in [1]. In the case of equal masses $m_1 = m_2 = m$ $(g' = 1)$ the expression (19) moves over to the WKB quantization condition of levels energy that was obtained in the case of spin particles with the equal masses in [20].

3. The relativistic leptonic decay widths in WKB approximation

Within the RQP approach, the relativistic modification of nonrelativistic expression for the leptonic decay widths of vector mesons as a bound system of two spin quarks of equal mass m in the state of energy M_n with relative orbital moment $\ell = 0$, proposed by Matveev, Struminskii, and Tavkhelidze in [21] (or by Van Royen and Weisskopf in [22], see also Refs. [23–25]), can be represented through solution (13) in the form [2, 3]

$$\Gamma_{n,\ell=0}(V \to e^+e^-) = \frac{16\pi\alpha^2 Q_V^2 f_1^2(t)}{M_n^2} \lim_{r \to i\lambda'} \left| e^{-\pi\tilde{\rho}'/2} \Gamma(1 + i\tilde{\rho}') \frac{\varphi_0^L(r, \chi_n)}{r} \right|^2. \tag{20}$$

Here

$$\tilde{\rho}' = \frac{\tilde{\alpha}'_s a' \cosh \chi'}{4}, \; \tilde{\alpha}'_s = \frac{\alpha_s}{\lambda'},$$

$f_1(t)$ is the form factor of quark in expression of the current operator of a quark, as function of the square of the 4-momentum transfer

$$t = (p' - p)^2 = 2M_Q^2 c^4 (1 - \cosh \chi_\Delta), \tag{21}$$

where p, p' are the initial and final momentums and χ_Δ is the corresponding rapidity. The form factor $f_1(t)$ depends on the anomalous magnetic moment of quark μ_{anom} and into the nonrelativistic case in Ref. [22] μ_{anom} was taken equal to zero, and, hence, the form factor $f_1(t)$ was chosen equal to unit. In the case of relativistic quarks the form factor $f_1(t)$ we choose in the manner of

$$f_1(t) = \frac{\sinh \chi_\Delta}{\chi_\Delta}, \tag{22}$$

where $\chi_\Delta / \sinh \chi_\Delta$ is the relativistic geometric factor in the RQP approach [7], which for the first time validly came up in Ref. [26] for invariant description of the spatial structure of the particles in the three-dimensional relativistic $\mathbf{r}$ representation [8] and it serves the contribution measure of relativistic effects of quarks.

We note that the additional factor $\exp(-\pi\tilde{\rho}'/2)\Gamma(1 + i\tilde{\rho}')$ ensures a correct relativistic limit equal to unit for the threshold resummation S factor for the composite system of two relativistic arbitrary-masses of spin quarks for $\chi' \to +\infty$ $(u \to 1)$ and the transition to the spinless case at $a' = 0$ and $b' = 2/g'm'c^2$ (for more details, see [17]). We see also that, in the spinor case, the function

$$\psi_0(r,\chi) = e^{-\pi\tilde{\rho}/2}\Gamma(1 + i\tilde{\rho})\varphi_0^L(r,\chi)$$

is the physical wave function of s-wave state of the composite system that consistes of two relativistic spinor particles of arbitrary-masses, interacting by means of potential (1).

For potential (1) the WKB approximation of the RQP radial wave function $\varphi_\ell^L(r,\chi_n)$ in region $r \in (r_L; r_R)$ in accordance with Eqs. (13) and (14) can be presented in the form

$$\varphi_\ell^L(r,\chi_n) = \frac{C_\ell(\chi_n)}{\sqrt[4]{[\mathcal{X}^2(r) - R^2(r)][1 + a'V(r)X(r)]}} \sin\left\{\frac{1}{\lambda'}\int\limits_{r_L}^{r} dr'\Big[\chi_+(r') - \ln R(r')\Big]\right\}, \tag{23}$$

where the normalization constant $C_\ell(\chi_n)$ is found from the normalization condition

$$4\pi \int\limits_0^\infty dr \left|\varphi_\ell^L(r,\chi_n)\right|^2 = 1, \; \ell \geq 0. \tag{24}$$

In the region of applicability of the relativistic WKB method the argument of sine in (23) is the quickly oscillating function. Therefore, the square of sine in (24) can be replaced, either as in the nonrelativistic case, on its the average importance equal 1/2 [1–3]. Instead of Eq. (24) we then obtain

$$2\pi|C_\ell(\chi_n)|^2 \int\limits_{r_L}^{r_R} \frac{dr}{\sqrt{[\mathcal{X}^2(r) - R^2(r)][1 + a'V(r)X(r)]}} = 1. \tag{25}$$

Differentiation of the WKB quantization condition of energy levels (19) on the total energy $M_n = 2g'm'c^2\cosh\chi_n$ under $\ell \geq 0$, where potential $V_{\text{conf}}(r)$ does not depend on energy M_n, and taking into consideration determinations (14) and condition (15) for the turning points $r_{L,R}$, we come to the condition

$$\int\limits_{r_L}^{r_R} \frac{dr}{\sqrt{[\mathcal{X}^2(r) - R^2(r)][1 + a'V(r)X(r)]}} = 2\pi\lambda' g'm'c^2 \frac{dn}{dM_n}. \tag{26}$$

From Eqs. (25) and (26) we find

$$|C_\ell(\chi_n)|^2 = \frac{1}{4\pi^2 \lambda' g' m' c^2} \frac{dM_n}{dn}. \tag{27}$$

Then, either as in the Refs. [1–3, 27], the WKB radial wave function (23) of the potential (1) at $\ell = 0$ for enough of the large value of $\rho = r/\lambda', r \in (r_L; r_R)$, but such, where the Coulomb interaction $V_{\text{Coul}} = -\alpha_s/r$ will dominate in the potential (1), can be approximated by the Coulomb radial s-wave function for which its the exact form is the known [17]

$$\begin{aligned} \varphi_0^{\text{Coul}}(\rho, \chi') = {} & 2\pi C_0^{\text{Coul}}(\chi') e^{iB'\chi' - \chi' + i(\rho - \tilde{\rho}')\chi'} (\rho - \tilde{\rho}') \times \\ & \times F\big(1 - iB', 1 - i(\rho - \tilde{\rho}'); 2; 1 - e^{-2\chi'}\big). \end{aligned} \tag{28}$$

Here $F(a, b; c; z)$ is the hypergeometric function, $C_0^{\text{Coul}}(\chi')$ is the normalization constant, the parameter B' is defined as

$$B' = \frac{\tilde{\alpha}'_s(a' \cosh^2 \chi' + b')}{4 \sinh \chi'}, \tag{29}$$

and at $\chi' = i\kappa_n$ the parametr B' is connected with the quantization condition of the energy levels for the Coulomb potential by expression [17]

$$\frac{\tilde{\alpha}'_s(a' \cos^2 \kappa_n + b')}{4 \sin \kappa_n} = n, \ \ell = 0, \ n = 1, 2, \dots, \ 0 < \kappa_n < \pi/2.$$

Comparing of asymptotic expression for the Coulomb wave function in (28),

$$\varphi_0^{\text{Coul}}(\rho, \chi')\big|_{\rho \gg 1} \sim \frac{2\pi C_0^{\text{Coul}}(\chi') e^{-\pi B'/2}}{\sinh \chi' |\Gamma(1 - iB')|} \sin\left[\rho\chi' + \delta_0^{\text{Coul},S}(\chi')\right],$$

with the asymptotic form of WKB solution in (23), taken at $\ell = 0$,

$$\varphi_0^L(\rho, \chi')\big|_{\rho \gg 1} \sim \frac{C_0(\chi')}{\sqrt{\sinh \chi'}} \sin\left[\rho\chi' + \delta_0^{\text{Coul,WKB},S}(\chi')\right],$$

gives the relationship between the normalization constants

$$\left|2\pi C_0^{\text{Coul}}(\chi')\right|^2 = \sinh \chi' e^{\pi B'} |\Gamma(1 - iB')|^2 \left|C_0(\chi')\right|^2, \tag{30}$$

where

$$\delta_0^{\text{Coul},S}(\chi') = B' \ln(2\rho \sinh \chi') - \tilde{\rho}'\chi' + \arg\Gamma(1 - iB')$$

is the phase of the Coulomb wave function in (28), and

$$\delta_0^{\text{Coul,WKB},S}(\chi') = B' \ln\left(\frac{2\rho \sinh \chi'}{B'}\right) - \tilde{\rho}'\chi' \tag{31}$$

is its expression in the WKB approximation [19].

Finally, taking into consideration Eqs. (20), (27), (28), and (30), we get expression for the relativistic leptonic decay widths of the vector mesons in s-wave state ($\ell = 0$) and with energy M_n as the composite system of two relativistic spinor quarks of arbitrary-masses, interacting by means of potential (1):

$$\Gamma_{n,\ell=0}(V \to e^+e^-) = \frac{4\alpha^2 Q_V^2 f_1^2(t)(\hbar c)^3 \sinh\chi_n}{\pi\lambda'^3 g' m' c^2 M_n^2} S_{\mathrm{RQP,S}}(\chi_n)\frac{dM_n}{dn}. \tag{32}$$

Here

$$S_{\mathrm{RQP,S}}(\chi') = \frac{X_{\mathrm{RQP,S}}(\chi')}{1-\exp\left[-X_{\mathrm{RQP,S}}(\chi')\right]} e^{-\pi\tilde{\rho}'}\left|\Gamma(2+i\tilde{\rho}')F(1+iB',-i\tilde{\rho}';2;1-e^{-2\chi'})\right|^2 \tag{33}$$

is the resummation Coulomb S factor for the composite system of two relativistic spin particles of arbitrary-masses m_1, m_2, that interacts by means of the Coulomb potential, which appears in considered RQP approach[3], where the value $X_{\mathrm{RQP,S}}(\chi')$ is connected with parameter B' in (29) by expression

$$X_{\mathrm{RQP,S}}(\chi') = 2\pi B' = \frac{\pi\tilde{\alpha}'_s(a'\cosh^2\chi' + b')}{2\sinh\chi'}, \tag{34}$$

which can be presented by means of (8) in term of the velocities (4) and (5) in the form

$$X_{\mathrm{RQP,S}}(u) = \frac{\pi\tilde{\alpha}'_s\sqrt{1-u^2}}{2g'u}\left[g'^2(a'+b') + \frac{a'u^2}{1-u^2}\right] = \frac{\pi\tilde{\alpha}'_s}{g'u'_{\mathrm{rel}}}\left[g'^2(a'+b') + \frac{a'}{4}u'^2_{\mathrm{rel}}\right]. \tag{35}$$

We note that at $a' = 0, b' = 2/g'm'c^2$ Eq. (32) moves over to Eq. (2) for the case spinless, taken at $\ell = 0, Q_V = e_q, f_1(t) = 1$.

4. The study of influence spin parameters of vector mesons on behavior of their leptonic decay widths

We shall conduct the study of the influence spin parameters a' and b' in (10) on importances of the relativistic leptonic decay widths for vector mesons in the s-wave state. On Fig. 1, we present the behavior of the function $R = R(u)$, as a function of the velocity u, Eq. (5). The function $R = R(u)$ is defined as the ratio of the relativistic leptonic decay widths of s-state vector meson, that is presented by Eqs. (32), (33) and (35), to its the relativistic spinless of analogue, presented by (2) and (7), taken at $\ell = 0, f_1(t) = 1, Q_V = e_q, \hbar = c = 1, \tilde{\alpha}_s = \tilde{\alpha}'_s = \alpha_s = 0.2$. The curves on Fig. 1 correspond the different importances of the factor g': the solid curve corresponds to $g' = 1$, while the dashed and the dot-dashed curves correspond to $g' = 1.5$ and $g' = 2.5$, respectively; the dotted line answers to importance $\alpha_s = 0$.

From Fig. 1 we see that in the nonrelativistic region of the velocity values of u ($u \ll 1$) the spin and masses of quarks (the factor g'), which form the spinor parameters a' and b' of vector mesons, affects substantially the behavior of the functions $R = R(u)$ and, hence, on the behavior of the leptonic decay widths of vector mesons. When the velocity u grows, the influence of the spinor parameters a' and b' of vector mesons on the behavior of function $R = R(u)$ becomes weaker, and $R \to 1$ in the relativistic limit ($u \to 1$),

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[3] We shall remind that the S factor in (33) reproduces both the known nonrelativistic limit ($v \to 1$) in the case spinless ($m_1 = m_2 = m, a' = a = 0, b' = b = 2/mc^2$), and the expected relativistic ($u \to 1$) and ultrarelativistic ($m' \to 0$) limits for the importance of parameters a' and b' in (10) (for details, see Refs.[17, 28–30]).

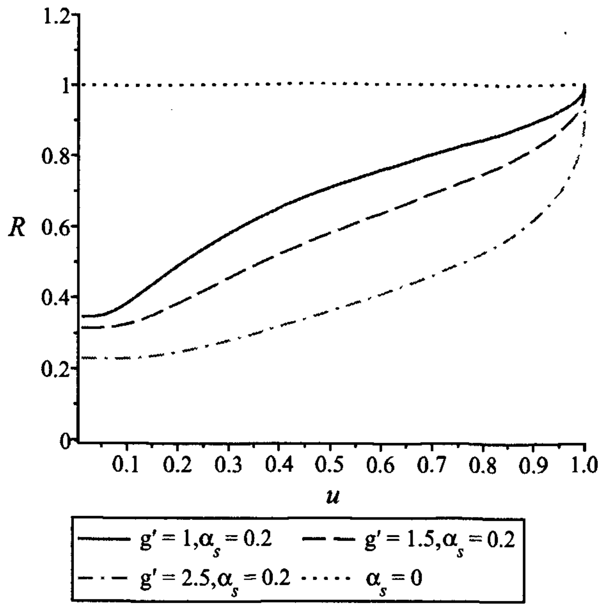

Figure 1. Ratio $R = R(u)$ of the relativistic leptonic decay widths for the s-state of vector meson, that is presented by Eqs. (32), (33) and (35), to its the relativistic spinless of analogue, presented by Eqs. (2) and (7), taken at $\ell = 0, f_1(t) = 1, Q_V = e_q, \hbar = c = 1, \tilde{\alpha}_s = \tilde{\alpha}'_s = \alpha_s = 0.2$. The solid curve corresponds to $g' = 1$, the dashed and the dot-dashed curves correspond to $g' = 1.5$ and $g' = 2.5$, respectively; the dotted line answers to importance $\alpha_s = 0$.

that is, this the influence disappears. Also, Fig. 1 show that the behavior of the function $R = R(u)$ depends on the factor g' in broad region of importances of the velocity u, and this dependence becomes weaker in the relativistic region of importances of the velocity u ($R \to 1$ at $u \to 1$).

As an example of the application of Eq. (32), on Fig. 2 we presented the behavior of the relativistic leptonic decay widths for the ρ_0-, ω-, and ϕ-mesons in the s-wave state and level $n = 1$, as a functions of the rapidity $\chi' = \chi$. The quarks of vector mesons interact by means of the linear potential with Coulomb-like (chromodynamical) potential,

$$V(r) = \sigma r - \frac{\alpha_s}{r}.$$

The curves on Fig. 2 correspond the different importances of the parameters for the ρ_0-, ω-, and ϕ-mesons from the Tabl. 1.

From Fig. 2 is seen that in the small region of the rapidity values of χ ($\chi \le 0.3$) the leptonic decay widths importances of Γ for the ground level of s-wave state of vector ρ_0, ω and ϕ-mesons differ small. With growing of the rapidity values of χ the differences in behavior of the leptonic decay widths importances of Γ for the ground level of s-wave state of vector ρ_0, ω and ϕ-mesons become essential.

Masses for the ρ_0-, ω- and ϕ-mesons were chosen equals [31]: $M_{\rho_0} = 775.26\,\text{MeV}$, $M_\omega = 782.65\,\text{MeV}$ and $M_\phi = 1019.461\,\text{MeV}$. Importances of the factor g', which is defined by formula (3) through relations of the masses m_u, m_d, m_s for the u-, d-, s-quarks, that form the ρ_0-, ω- and ϕ-mesons, were chosen equals: $g'_{\rho_0} = g'_\omega = 1.0012$ either as in Ref. [19] for $\pi^\pm$-meson, and $g'_\phi = 1$ because of equal masses. Importances of the masses m_u, m_d, m_s for the u-, d-, s-quarks were found in Ref. [19] at study of the spectrum of masses pseudoscalars of $\pi^\pm$-, $K^\pm$- and K_0-mesons: $m_u = 62.57\,\text{MeV}, m_d = 69.00\,\text{MeV}, m_s = 262.29\,\text{MeV}$. Importances of the rapidity χ', that corresponds importances of the masses M, m_u, m_d, m_s

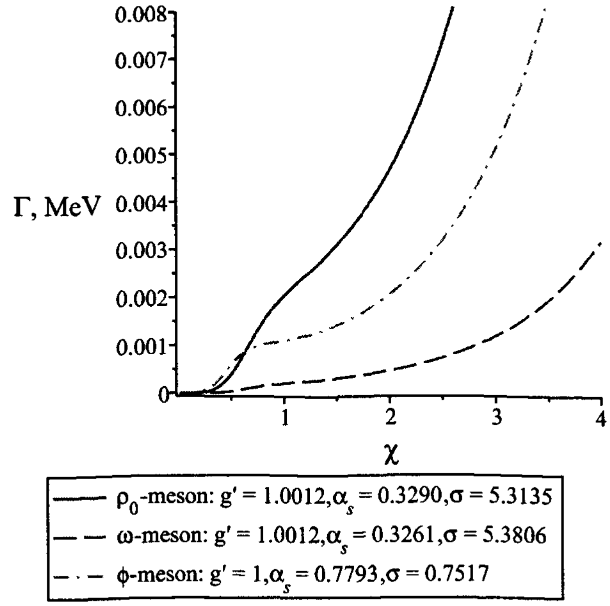

Figure 2. The relativistic leptonic decay widths, $\Gamma = \Gamma(\chi)$, for the ρ_0-, ω-, and ϕ-mesons in the s-wave state and level $n = 1$, that is presented by Eqs. (32), (33) and (34), as a functions of the rapidity χ, taken at $\hbar = c = 1$. The curves on Fig. 2 correspond the different importances of the parameters for the ρ_0-, ω-, and ϕ-mesons from the Tabl. 1. The solid curve corresponds to ρ_0-meson, the dashed and the dot-dashed curves correspond to ω-, and ϕ-mesons.

and the factor g' for the ρ_0-, ω- and ϕ-mesons, we find according of formulas (3) and (11) from expressions

$$M_{\rho_0} = 2m'_{\rho_0}g'_{\rho_0}\cosh\chi_{\rho_0},\ M_\omega = 2m'_\omega g'_\omega\cosh\chi_\omega,\ M_\phi = 2m'_\phi g'_\phi\cosh\chi_\phi.$$

Importances of the multiplier Q_V, conditioned by isotopic structure of vector mesons V and charges of quarks expressed in unit of the electric charge e, for the ρ_0-, ω- and ϕ-mesons were taken from Refs. [21, 22]: $Q_{\rho_0} = 1/\sqrt{2}, Q_\omega = 1/3\sqrt{2}, Q_\phi = -1/3$. By using the quantization conditions of the energy levels for vector mesons [19],

$$\frac{4}{a'\tilde{\sigma}'}\left[\frac{\cosh\chi'}{\sqrt{b'/a'}\sqrt{1+b'/a'}}\,\mathrm{arctanh}\left(\frac{\tanh\chi'\sqrt{b'/a'}}{\sqrt{1+b'/a'}}\right) - \right.$$
$$\left. -\frac{1}{\sqrt{1+b'/a'}}\arctan\left(\frac{\sinh\chi'}{\sqrt{1+b'/a'}}\right)\right] = \pi\left(n+\frac{\ell}{2}+\frac{3}{4}\right) - \delta_\ell^{\mathrm{Coul,WKB},S}(\chi'),$$
$$n = 0, 1, \ldots, \ \ell \geq 0,$$

we calculated the importances of linear ($\tilde{\sigma}' = \sigma$) and Coulomb ($\tilde{\alpha}'_s = \alpha_s$) of interaction constants and value of $dM_n/dn = 2m'g'\sinh\chi'_n d\chi'_n/dn$ for the ρ_0-, ω- and ϕ-mesons in s-state and level $n = 1$ across importances of the factor g' and rapidity χ' from Tabl. 1, where the phase of the Coulomb radial RQP wave function in the WKB approximation, $\delta_\ell^{\mathrm{Coul,WKB},S}(\chi')$ at $\ell = 0$, gives by Eq. (31) taken in the turning point $\rho_+ = 4(\cosh\chi' - 1)/\tilde{\sigma}'(a' + b')$, and the spin parameters a' and b' for vector mesons are given in (10). Importances of the rapidity χ_Δ for form factor $f_1(t)$ in (22) for the ρ_0-, ω- and ϕ-mesons were chosen equals: $\chi_{\Delta_{\rho_0}} = 3.2042$, $\chi_{\Delta_\omega} = 3.0081$ and $\chi_{\Delta_\phi} = 2.4054$. This importances of the rapidity χ_Δ correspond to importances of the relativistic leptonic decay widths for the ρ_0-, ω-, and ϕ-mesons in the s-wave state and level $n = 1$ from Tabl. 1 [31].

Table 1: Importances of leptonic decay widths and parameters for the ρ_0-, ω- and ϕ-mesons

Mesons	M, MeV	m_u, MeV	m_d, MeV	m_s, MeV	m', MeV	g'	χ'	Q_V	σ	α_s
ρ_0	775.26	62.57	69.00		65.71	1.0012	2.4595	$1/\sqrt{2}$	5.3135	0.3290
ω	782.65	62.57	69.00		65.71	1.0012	2.4691	$1/3\sqrt{2}$	5.3806	0.3261
ϕ	1019.461			262.29	262.29	1	1.2836	-1/3	0.7517	0.7793

Mesons	χ_Δ	$-t$, GeV^2	u	Γ_{theor}, keV	Γ_{exp}, keV
ρ_0	3.2042	13.63	0.9855	7.04	7.04 ± 0.06
ω	3.0081	11.21	0.9858	0.60	0.60 ± 0.02
ϕ	2.4054	9.53	0.8574	1.251	1.251 ± 0.021

From Tabl. 1 is seen that quarks, that form the ρ_0-, ω- and ϕ-mesons in s-state and level $n = 1$, are the relativistic ($u > 0.85$).

5. Conclusion

In the present study, the new relativistic expression for the leptonic decay widths of vector mesons in s-wave state have been obtained on the basis of the RQP approach in the relativistic semiclassical approximation. The present analysis has been performed for the case where relativistic spin quarks of arbitrary-masses that form vector mesons interact via a funnel-like potential including a purely confining part, which is not singular, and a singular part in the form of a Coulomb-like chromodynamical potential. For this aim the fully covariant finite-difference RQP equation in the three-dimensional relativistic $\mathbf{r}$ representation [8] for the case of interaction between two relativistic spin particles of arbitrary-masses has been solved by the relativistic WKB method. The condition of applicability of the WKB approximation has been established. It has been shown that, at $a' = 0$ and $b' = 2/g'm'c^2$, the new expression for the leptonic decay widths of vector mesons reduces to its relativistic spinless analog. The comparison of the new expression with its relativistic spinless analogue is executed. The influence of the spin and masses of quarks, which form the spin parameters a' and b' of vector mesons, on the behavior of leptonic decay widths of vector mesons in the s-wave state has been explored. This the influence of the spin and masses of quarks on the behavior of leptonic decay widths of vector mesons in the s-wave state is essential in the region of small values of the velocity u (it the so-called Sommerfeld effect), but when the velocity u grows, its influence becomes weaker, and in the relativistic limit ($u \to 1$) its the influence disappears. Importances of the relativistic leptonic decay widths for the ρ_0-, ω-, and ϕ-mesons in the s-wave state and level $n = 1$ were culculated. Since the new relativistic expression for the relativistic semiclassical leptonic decay widths of vector mesons has been obtained within a fully covariant method and has a correct connection with the Bethe—Salpeter function, one can expect that this expression takes into account more adequately both the relativistic character of interacting particles and their spin and masses.

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