Nonlinear Dynamics and Applications. Vol. 28 (2022) 431 - 439

The Mellin-Barnes technique and exact form of the three bubble-diagram contributions to the lepton anomaly

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Based on the Mellin–Barnes representation technique, new exact expressions are obtained for the contributions to the anomalous magnetic moment of a lepton L ($L = e$, μ and τ) from a class of Feynman diagrams with insertions of the vacuum polarization with three lepton loops. The corresponding analytical expressions are obtained as functions of mass ratio m_ℓ/m_L of loop leptons ℓ to the external one in the whole region $0 < m_\ell/m_L < \infty$. The obtained results are checked for consistency with known analytical expansions available in the literature.

PACS numbers: 13.40.Em, 12.20.Ds, 14.60.Ef
Keywords: anomalous magnetic moment of leptons, Mellin-Barnes representation, Feynman diagrams, electromagnetic vacuum-polarization contribution

1. Introduction

Our knowledge of the physics of elementary particles is based on comparisons of theoretical predictions with corresponding experimental data of processes involving elementary particles. The measurements of anomalous magnetic moments for both, electron a_e, and muon, a_μ, become more and more accurate. This serves as additional motivation to improve theoretical predictions, especially considering that there is a long-standing discrepancy between experimental data and theoretical predictions. This may indicate on limitations of the Standard Model (SM) in describing these data and probably, on the presence of new physics beyond the SM. There is a huge amount of articles and reports dedicated to the subject, see e.g., Refs. [1, 2] and references therein quoted.

Nowadays, the measured discrepancy for the electron anomaly $\Delta a_e \equiv a_e^{\rm exp} - a_e^{\rm SM} \simeq (-87 \pm 36) \times 10^{-14}$ exhibits a significance $\sim$ 2.5 standard deviation of the calculated

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anomaly from the experimental one (see, e.g., [3, 4] for details). The most recent measurement of the Muon ($g-2$) Collaboration by E989 experiment at Fermilab [5] combined with the previous results of the E821 experiment at the Brookhaven National Laboratory [6] point on $\Delta a_\mu = (251 \pm 59) \times 10^{-11}$ which implies that the discrepancy is ~ 4.2 standard deviation [5]. This circumstance, as well as the unexpected sign of Δa_e, opposite to Δa_μ, stimulates further theoretical investigations and search for reasons of possible deviations from the SM predictions. It is important also that in the near future this discrepancy will be further explored at Fermilab [7] and J-PARC [8] experiments. According to the SM, the contributions to a_L can be divided into the quantum electromagnetic (QED), electroweak and hadronic parts. Currently, calculations of the eighth- and tenth-order QED corrections to a_L, which are important in reduction of the theoretical uncertainties, are mainly performed numerically. Independent re-examinations of existing theoretical calculations of a_L can improve significantly the reliability of the quite cumbersome and computer time consuming numerical results.

Here we present an approach based on the well-known Mellin–Barnes integral transform of parametric Feynman integrals combined with dispersion relations, with the goal of obtaining exact analytical expressions for the higher order QED corrections, up to the eighth order, to the anomalous magnetic moment from the Feynman diagrams with vacuum polarization insertions of up to three lepton loops, as depicted in Fig. 1. The corresponding technique was presented in detail in Ref. [9], where it was applied to calculate explicitly radiative corrections up to the sixth order to the muon anomaly and asymptotical expansions for the eighth order corrections. In this paper we present a generalization of the method to be applied to any kind of leptons and to find analytical expressions to the lepton anomaly including also the eighth order.

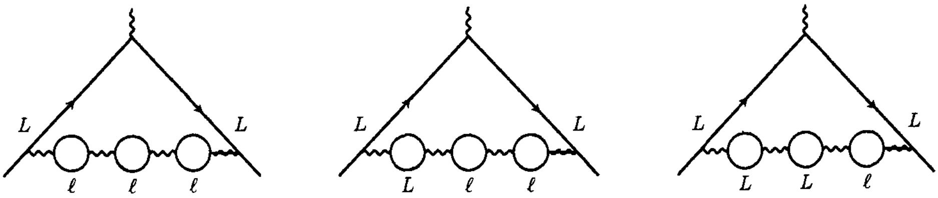

Figure 1: The 8-th order diagrams considered in the present paper. From left to right: three identical lepton loops ($\ell\ell\ell$), $l \neq L$, left panel; two identical lepton loops formed by leptons ℓ different from the external one, ($L\ell\ell$), central panel; two lepton loops formed by the same leptons as the external one, ($LL\ell$), right panel.

2. Basic formulae

As commonly adopted, the QED contributions to the anomalous magnetic moment a_L can be classified as (see, e.g., Ref. [1] for details)

$$a_L^{(\mathrm{QED})} = A_1 + A_2\left(\frac{m_{\ell_1}}{m_L}\right) + A_2\left(\frac{m_{\ell_2}}{m_L}\right) + A_3\left(\frac{m_{\ell_1}}{m_L}, \frac{m_{\ell_2}}{m_L}\right). \tag{1}$$

Each term in Eq. (1) can be represented as an expansion in the fine structure constant α. Obviously, the coefficients for A_1 are universal for all kinds of leptons. The leading order coefficient $A_1^{(2)} = 1/2$ was firstly obtained long ago by Schwinger [10]. The exact analytical results for the coefficients $A_1^{(4)}$ and $A_1^{(6)}$ are also available [11–13] (see also Refs. [2, 14]

for more details). Here we focus on the mass-dependent part A_2 which is given as

$$A_2 = A_2^{(4)} \left(\frac{\alpha}{\pi}\right)^2 + A_2^{(6)} \left(\frac{\alpha}{\pi}\right)^3 + A_2^{(8)} \left(\frac{\alpha}{\pi}\right)^4 + \cdots . \tag{2}$$

Evidently, the coefficient A_1 immediately follow from the explicit expressions for $A_2(m_\ell/m_L)$ as the limit $m_\ell/m_L \to 1$.

Among all possible diagrams contributing to A_2 we restrict ourselves to those which correspond to diagrams (the bubble-like diagrams) with photon vacuum polarization insertions, as depicted in Fig. 1. In what follows, for the coefficients in Eq. (2) corresponding to the considered diagrams we use the notation $\mathcal{A}_2^{(2n)}$. The exact analytic expression for $\mathcal{A}_2^{(4)}$ firstly has been reported in Ref. [16] and confirmed later in a variety of approaches, including the Mellin-Barnes transform, cf. Refs. [17–20]. The exact form for the coefficient $\mathcal{A}_2^{(6)}$ was obtained for the first time in Ref. [21], see a brief history in Ref. [22]. The analytical expansions for $\mathcal{A}_2^{(8)}$ in terms of the lepton mass ratios and the numerical values are known (see, for example, Refs. [2, 9, 24–29]). Analytical expressions for $\mathcal{A}_2^{(8)}$ are not yet known in the literature. However, knowledge of the corresponding exact formulae is of a great importance for, it presents a serious test of the previously reported results, e.g., test for asymptotic formulas and numerical calculations. The main purpose of this work is to find exact analytic forms for the coefficients $\mathcal{A}_2^{(8)}$ using the dispersion relations and the Mellin–Barnes technique for the Feynman parametrization of the corresponding diagrams.

The use of dispersion relations for the vacuum polarization insertions in the considered Feynman diagrams leads to an expression in which the dressed photon propagator is formally replaced by a propagator of a virtual massive photon. Then, further calculations are based on the Mellin-Barnes representation for propagator functions, cf. Refs. [19, 20, 30–32].

$$\frac{1}{(1+X)^\beta} = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} (X)^{-z} \frac{\Gamma(z)\Gamma(\beta - z)}{\Gamma(\beta)}\, dz . \tag{3}$$

As mentioned in Ref. [19], the application of Eq. (3) together with the converse mapping theorem [33], which relates the asymptotic behavior with the singularities of the integrand, provide an effective way to obtain analytically asymptotics of the Feynman diagrams in terms of a mass ratio.

The Mellin-Barnes representation for 8-th order coefficients $\mathcal{A}_2^{(8)}$ corresponding to diagrams shown in Fig. 1 are given as

$$\mathcal{A}_2^{(8),\ell\ell\ell} = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} dz\, (4t)^{-z}\, \Gamma(z)\Gamma(1-z)\, \Omega_0(z)\, R_3(z) , \tag{4}$$

$$\mathcal{A}_2^{(8),L\ell\ell} = \frac{3}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} dz\, (4t)^{-z}\, \Gamma(z)\Gamma(1-z)\, \Omega_1(z)\, R_2(z) , \tag{5}$$

$$\mathcal{A}_2^{(8),LL\ell} = \frac{3}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} dz\, (4t)^{-z}\, \Gamma(z)\Gamma(1-z)\, \Omega_2(z)\, R_1(z) , \tag{6}$$

where t denotes the ratio of squared masses:

$$t = \frac{m_\ell^2}{m_L^2} . \tag{7}$$

The expressions for $\Omega_{0,1,2}(z)$ and $R_{1,2,3}(z)$ can be found in Refs. [9, 34].

3. Analytic result

With explicit expressions for $\Omega(s)$ and $R(s)$, the integrations in Eqs. (4)-(6) can be performed by the Cauchy's residue theorem. Closing the integration contour in the left hemisphere, for $t < 1$, or in the right one, for $t > 1$, we obtain the desired analytical expressions.

3.1. Three identical closed lepton loops

In the case of three identical lepton loops, the integral (4) reads as

$$\mathcal{A}_2^{(8),\ell\ell\ell}(t) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} t^{-z}\left[\frac{(1-z)}{(z+2)(2z+1)(2z+3)(2z+5)(2z+7)(2z+9)}\right]\times \tag{8}$$
$$\left[\frac{-(1-z)(-3492+5256z+31831z^2+41045z^3+22650z^4+5632z^5+512z^6)}{27z(z+1)^2(z+2)}\right.$$
$$\left. -\pi^2(35+21z+3z^2)+\ 6\psi^{(1)}(z)(35+21z+3z^2)\right]\frac{\pi^2}{\sin^2(\pi z)}\,dz$$

where $\psi^{(1)}(z)$ is the polygamma function of the first order.

The result of integration we present in a series of powers of $\ln(t)$

- $\mathcal{A}_2^{(8),\ell\ell\ell}(t<1) = C_0(t) + C_1(t)\ln t + C_2(t)\ln^2 t + C_3(t)\ln^3 t + C_4(t)\ln^4 t + \Sigma_1(t)\,,$ (9)

$$C_0 = -\frac{8609}{5832} - \frac{25\pi^2}{162} + \left(\frac{1760}{567}+\frac{26\pi^2}{27}\right)t + \left(\frac{13786139}{793800}+\frac{127\pi^2}{162}+\frac{\pi^4}{5}\right)t^2 - \left(\frac{183313181}{44651250} - \right.$$
$$\left.\frac{224\pi^2}{225}\right)t^3 - \left(\frac{145307}{63000}+\frac{1051\pi^2}{2880}\right)t^4 - \left(\frac{631}{1050}-\frac{133\pi^2}{2880}\right)t^5 - \left(\frac{499}{4200}-\frac{7\pi^2}{800}\right)t^6 - \frac{\pi^4}{1536}$$
$$\times\left(q_1(t)+11t^4\right)\sqrt{t} - \left(\frac{1292159}{12700800}-\frac{11\pi^2}{1536}\right)t^4\Phi\left(t,2,-\frac{1}{2}\right) + \frac{11}{512}\,t^4\Phi\left(t,4,-\frac{1}{2}\right)$$
$$-\frac{1}{2}\left(q_3(t)-\frac{\pi^2}{768}q_1(t)\right)t^4\Phi\left(t,2,\frac{7}{2}\right) + \frac{1}{512}\,q_1(t)\,t^4\Phi\left(t,4,\frac{7}{2}\right) + q_2(t)\,\mathrm{Li}_2(t)$$
$$-\frac{32}{315}\,t\,(-12+7t)\,\mathrm{Li}_3(t) - 2\,t^2\mathrm{Li}_4(t) + \left(-\frac{2}{9}+\frac{136}{35}t-\frac{64}{15}t^2+\frac{16}{15}t^3\right)\zeta_3\,; \tag{10}$$

$$C_1 = -\frac{317}{324} - \frac{\pi^2}{27} + \left(\frac{1301567}{181440}+\frac{721\pi^2}{2304}\right)t + \left(-\frac{77341897}{6350400}+\frac{2359\pi^2}{2304}\right)t^2 + \left(-\frac{314708203}{95256000}\right.$$
$$\left.+\frac{263\pi^2}{2304}\right)t^3 \ + \left(-\frac{441989}{254016}+\frac{11\pi^2}{768}\right)\,t^4 - \frac{539}{7200}\,t^5 - \frac{21}{2000}\,t^6 + \Big[\ q_3(t) + \frac{1292159}{6350400}t^4 - \frac{\pi^2}{768}$$
$$\times\left(q_1(t)+11t^4\right)\Big]\sqrt{t}\arctan\left(\sqrt{t}\right) - \frac{11}{512}\,t^4\Phi\left(t,3,-\frac{1}{2}\right) - \frac{1}{512}q_1(t)\,t^4\Phi\left(t,3,\frac{7}{2}\right)$$
$$+\ q_2(t)\ln(1-t) + \frac{16}{315}\,(-12+7t)\,t\,\mathrm{Li}_2(t) + 2\,t^2\mathrm{Li}_3(t) - \left(\frac{236}{27}+\frac{16\pi^2}{27}-4\,\zeta_3\right)t^2\,; \tag{11}$$

(12)

$$C_2 = -\frac{25}{108} + \frac{13}{9}\,t + \left(\frac{127}{108} + \frac{\pi^2}{3}\right) t^2 + \frac{112}{75}t^3 + \frac{1051}{1920}t^4 \tag{13}$$
$$+ \frac{133}{1920}t^5 + \frac{21}{1600}t^6 + \frac{1}{1024}\,t^4 \left[11\Phi\left(t,2,-\frac{1}{2}\right) + q_1(t)\,\Phi\left(t,2,\frac{7}{2}\right)\right] - t^2\mathrm{Li}_2(t)\,;$$

$$C_3 = -\frac{1}{54} + \frac{721}{4608}\,t + \frac{2981}{13824}\,t^2 + \frac{263}{4608}\,t^3 + \frac{11}{1536}\,t^4 - \frac{1}{1536}\left(q_1(t) + 11t^4\right)\sqrt{t} \tag{14}$$
$$\times \arctan\left(\sqrt{t}\right) - \frac{1}{3}\,t^2\ln(1-t)\,;$$

$$C_4 = \frac{1}{12}\,t^2\,; \tag{15}$$

$$\Sigma_1(t) = 6\sum_{n=4}^{\infty}\left[X_3(n)\,H_n^{(2)} + X_4(n)\psi_{n+1}^{(2)} - X_4(n)H_n^{(2)}\ln t\right] t^n\,, \tag{16}$$

$$X_3(n) = \left(-295995 + 836500\,n - 787336\,n^2 + 206366\,n^3 + 131386\,n^4 - 114304\,n^5\right.$$
$$\left.+35216\,n^6 - 5088\,n^7 + 288\,n^8\right)/[Y_2(n)]^2\,,$$

$$X_4(n) = (1+n)\,(35 - 21\,n + 3\,n^2)/Y_2(n)\,,$$

$$Y_2(n) = (n-2)\,(2n-9)\,(2n-7)\,(2n-5)\,(2n-3)\,(2n-1)\,.$$

- $$\mathcal{A}_2^{(8),\ell\ell\ell}(t>1) = D_0(t) + D_1(t)\ln t + \Sigma_2(t)\,, \tag{17}$$

$$D_0(t) = q_5(t) - \frac{1}{2t^6}\left(-q_3(t) - \frac{1292159}{6350400}t^4\right)\Phi\left(\frac{1}{t},2,\frac{13}{2}\right) + \left(q_2(t) - \frac{2\pi^2}{3}t^2\right)\mathrm{Li}_2\left(\frac{1}{t}\right) +$$
$$\left(-\frac{32}{45}t^2 + \frac{128}{105}t\right)\mathrm{Li}_3\left(\frac{1}{t}\right)\,; \tag{18}$$

$$D_1(t) = q_4(t) - \left(q_3(t) + \frac{1292159}{6350400}\,t^4\right)\sqrt{t}\arctan\left(\frac{1}{\sqrt{t}}\right) + \left(q_2(t) + \frac{2\pi^2}{3}t^2\right)\ln\left(1-\frac{1}{t}\right)$$
$$-\left(\frac{64}{105} - \frac{16}{45}t\right)t\,\mathrm{Li}_2\left(\frac{1}{t}\right)\,; \tag{19}$$

$$\Sigma_2(t) = \sum_{n=2}^{\infty}\left[X_3(-n)\,H_{n-1}^{(2)} - X_4(-n)\psi_n^{(2)} - X_4(-n)H_{n-1}^{(2)}\ln t\right] t^{-n} \tag{20}$$

with

$$q_1(t) = -101 + 820\,t + 210\,t^2 + 84\,t^3\,,$$
$$q_2(t) = -\frac{97}{2835} + \frac{10348}{11025}\,t + \frac{5524}{675}\,t^2 - \frac{2\pi^2}{3}t^2$$
$$q_3(t) = -\frac{101}{64} + \frac{1025}{72}\,t + \frac{1813}{480}\,t^2 + \frac{3229}{2100}\,t^3\,,$$

$$q_4(t) = -\frac{1292159}{6350400}\,t^4 - \frac{30585647}{19051200}\,t^3 - \frac{137496269}{31752000}\,t^2 - \frac{1620853897}{222264000}\,t + \frac{307987}{893025}\,,$$

$$q_5(t) = \frac{6004}{675}\,t + \frac{61333}{33075} + \frac{58564}{59535\,t} + \frac{661005811}{1152597600\,t^2} + \frac{241718237}{691558560\,t^3} + \frac{61579}{313632\,t^4} - \frac{101}{3872\,t^5}\,,$$
$$q_6(t) = -\frac{32}{45}t^2 + \frac{128}{105}t\,.$$

Functions Φ, Li_n and H_n are the Lerch function, the Harmonic sum and the Polylogarithm function, respectively.

3.2. Two identical lepton loops formed by leptons different from the external one

For the case the corresponding expression can be represented by a single analytic function defined in the whole domain $t \in (0, \infty)$.

- $\mathcal{A}_2^{(8),\ell\ell L}(t) = F_0(t) + F_1(t)\ln t + F_2(t)\ln^2 t + F_3(t)\ln^3 t\,, \qquad (21)$

$$\begin{aligned} F_0 = {} & \frac{7627}{1944} + \frac{175}{18}\,t - \frac{54346}{151875}\,t^2 + \frac{31168}{13505625}\,t^3 - \frac{32}{15435}\,t^4 - \frac{4\pi^4}{45}\left(1+2t^2\right) \qquad (22) \\ & - \frac{\pi^2}{3}\,f_1(t) + \left(\frac{12}{35} - \frac{4}{45}t\right) t^4\, \Phi\left(t,3,\frac{9}{2}\right) + f_1(t)\,\mathrm{Li}_2(t) + f_2(t)\,\mathrm{Li}_3(t) \\ & + 4\left(1+2t^2\right)\mathrm{Li}_4(t)\,; \end{aligned}$$

$$\begin{aligned} F_1 = {} & \frac{61}{162} - \frac{\pi^2}{27} + \frac{136}{27}\,t - \frac{4\pi^2}{9}\,t - \frac{3734}{10125}\,t^2 + \frac{13\pi^2}{27}\,t^2 - \frac{5312}{385875}\,t^3 + \frac{16}{2205}\,t^4 \qquad (23) \\ & - \left(\frac{12}{35} - \frac{4}{45}t\right) t^4\, \Phi\left(t,2,\frac{9}{2}\right) + f_1(t)\,\ln(1-t) - f_3(t)\,\mathrm{Li}_2(t) - 2\left(1+2\,t^2\right)\mathrm{Li}_3(t)\,; \end{aligned}$$

$$\begin{aligned} F_2 = {} & \frac{2869}{3780} - \frac{29}{70}\,t + \frac{2081}{1890}\,t^2 + \frac{1}{27}\,t^3 - \frac{\pi^2}{9}\left(1+2t^2\right) + \left(\frac{12}{35} - \frac{4}{45}t\right)\frac{1}{\sqrt{t}}\,\mathrm{arctanh}\left(\sqrt{t}\right) \\ & - f_4(t)\ln\left(1-t\right) + \frac{1}{3}\left(1+2\,t^2\right)\mathrm{Li}_2(t)\,, \qquad (24) \end{aligned}$$

$$F_3 = -\frac{4}{45}\,t^2 + \frac{44}{945}\,t^3 \qquad (25)$$

with

$$\begin{aligned} f_1(t) &= -\frac{13}{9} + 4t - \frac{67}{27}\,t^2 - \frac{2}{27}\,t^3\,, \\ f_2(t) &= \frac{1}{3} + 4t - \frac{73}{15}\,t^2 + \frac{88}{315}\,t^3\,, \\ f_3(t) &= \frac{2}{9} + \frac{8}{3}\,t - \frac{154}{45}\,t^2 + \frac{88}{315}\,t^3. \end{aligned}$$

3.3. Two lepton loops formed by the same leptons as the external one

The coefficients $\mathcal{A}_2^{(8),\ell LL}$ have been calculated in the same manner as the previous one. Since the final expression is too cumbersome, we do not present it here and postpone for a more detailed paper. Nevertheless, the asymptotic expression $t \gg 1$ is more readable and, for comparison with other known results , we present it below

$$\begin{aligned} \mathcal{A}_2^{(8),\ell LL}(t) = {} & -\left(\frac{203}{486} - \frac{16}{45}\zeta(3)\right)\frac{1}{t} - \Bigg(\frac{40783}{4630500}\ln(t) + \frac{37}{44100}\ln^2(t) + \frac{1}{1260}\ln^3(t) \qquad (26) \\ & + \frac{1023526159}{5186160000} - \frac{17}{105}\zeta(3)\Bigg)\frac{1}{t^2} - \Bigg(\frac{1243103}{187535250}\ln(t) - \frac{1061}{595350}\ln^2(t) + \frac{2}{2835}\ln^3(t) \\ & - \frac{8}{945}\zeta(3) + \frac{4744350631}{472588830000}\Bigg)\frac{1}{t^3} - \Bigg(\frac{1013327141}{199687534200}\ln(t) - \frac{166657}{57629880}\ln^2(t) + \frac{2}{6237}\ln^3(t) \\ & - \frac{8}{2079}\zeta(3) + \frac{8721404003611}{2767669224012000}\Bigg)\frac{1}{t^4} - \frac{977387988901}{182797296932250}\ln(t) - \frac{4914517}{2029052025}\ln^2(t) \end{aligned}$$

$$+\left(\frac{16}{135135}\ln^3(t)+\frac{23356591160482123}{1646820848062640250 0}-\frac{64\zeta(3)}{45045}\right)\frac{1}{t^5}+O\left(\frac{1}{t^6}\right).$$

Other limiting cases ($t \ll 1$, and $t \to 1$) can also be easily obtained from the main analytical expression. So, as mentioned, the limit $t \to 1$ of $\mathcal{A}_2^{(8),\ell LL}(t \to 1)$ defines the coefficient A_1 in (1), $\mathcal{A}_2^{(8),\ell LL}(t=1) = 3\mathcal{A}_1^{(8)}$, where $\mathcal{A}_1^{(8)} = \dfrac{151849}{40824} - \dfrac{2\pi^4}{45} + \dfrac{32\zeta(3)}{63}$. It should be noted that all our limiting cases exactly coincide with the ones previously reported in the literature.

3.4. Discussion

Here below we compare our results with the known analytical expansions. Our expansions for $\mathcal{A}_2^{(8),\ell\ell\ell}$, $\mathcal{A}_2^{(8),\ell\ell L}$ and $\mathcal{A}_2^{(8),\ell LL}$, obtained for case $t < 1$, reproduce exactly the corresponding expressions reported in Ref. [9], cf. Eqs. (A1), (A3), and (A5), for $t > 1$ the expansions in Ref. [28]. In Fig. 2 our expansions are presented by dotted and short-dashed curves.

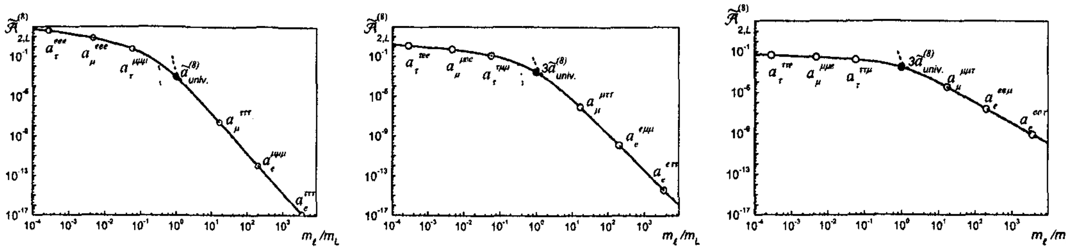

Figure 2: The behavior of 8-th order coefficient $\mathcal{A}_2^{(8)}$ vs. the mass ratio m_ℓ/m_L. From left to right: for case ($\ell\ell\ell$), Eqs. (9) and (17) – left panel; ($\ell\ell\ell$) Eq. (21) – central panel; $\ell \neq L$ ($LL\ell$) – right panel. The dotted lines represent the asymptotic expressions for the exact analytical formulae.

Figure 2 shows the general behavior of the coefficient $\mathcal{A}_2^{(8)}(m_\ell/m_L)$ corresponding to the diagrams in Fig. 1. The solid line represents the exact result, the dotted line is the result of using the corresponding expansion for the case $m_\ell/m_L < 1$, and the short-dashed line – for the case $m_\ell/m_L > 1$. Open circles indicate $\mathcal{A}_2^{(8)}(m_\ell/m_L)$ values for the physical masses of leptons. For a better illustration of the combination of leptons corresponding to the physical masses in this figure, we additionally labeled the open circles by $a_L^{\ell_1\ell_2\ell_3}$. The closed circle corresponds to the universal value $\mathcal{A}_2^{(8)}(t=1)$.

4. Summary

The Mellin–Barnes representation technique combined with dispersion relations to the Feynman parametrization of diagrams has been, for the first time, applied to obtain analytical expressions for the radiative corrections from the three loops diagrams to the lepton anomaly. It has been shown that the new expressions reproduce all previously known expansions and fully agree with direct numerical calculations. We argue that, in spite of the obtained explicit expressions in the case $t < 1$ and $t > 1$ are, at first glance, quite different, they represent two branches of the same analytical function.

Acknowledgments

We warmly thank A. V. Sidorov for the some temporary cooperation in the earlier stages of this work. This research was supported by the Belarus–JINR Collaborative Grant.

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