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Semileptonic Decay of Mesons in Point Form of Poincaré-Invariant Quantum Mechanics

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Well-known that the form-factors of semileptonic decays of light pseudoscalar mesons are determined with high precision both experimentally and in various approaches and models. In particular, for decay $\pi^{\pm} \to \pi^0 \ell^{\pm} \nu_{\ell\mp}$ in the limit of isotopic invariance results are $f_+(0) = \sqrt{2}$ and $f_-(0) = 0$. However within the framework of quark models obtaining such results is a non-trivial task.

The paper presents the form-factor calculating method of semileptonic meson decays in model, based on the point form of Poincare-invariant quantum mechanics. As a result of the work the values of the form factors $f_+(q^2)$ and $f_-(q^2)$ for zero transmitted momentum are obtained.

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1. Introduction

The accuracy of modern detectors makes it possible not only to determine the parameters of the standard model with a high degree of accuracy but also to explore the effects of new physics. In this regard, research in the field of flavour physics becomes important: analysis of electroweak decays of mesons makes it possible to study the mechanisms of interaction of quarks inside hadrons and also obtain the exact values of the elements of Cabibbo-Kobayashi-Maskawa (CKM) matrix.

From the variety of experimental data, we point out the leptonic and semileptonic decays of hadrons: to the present moment sufficient amount of information has been accumulated on such decays for both light and heavy mesons sector. Of particular interest are mesons with one or two light quarks since for the elements of the matrix CKM it becomes possible to study the relationship $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 + \delta_{CKM}$ [1], where δ_{CKM} – possible deviations. Definition of values V_{uj} $(j = b, s, d)$ carried out from leptonic and semileptonic decays of pseudoscalar $\pi^{\pm}$,$K^{\pm}$- and $B^{\pm}$-meson in conjunction with form-factors $f_{\pm}(q^2)$.

Theoretical studies of such decays are complicated by the fact that quarks are not observed in a free state. In this regard, a sufficient variety of approaches and models for calculating the characteristics of quarks inside hadrons has been developed. Among the

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variety of such models, we note the lattice models [2], which use the current masses of quarks. However, calculations in lattice models are based on $SU(3)$ symmetry with the assumption $m^2_{K^\pm} = m^2_{\pi^\pm}$ [3] which contradicts modern experimental data [4].

In group-theoretic approaches we marked out models based on the Poincaré group [5]. Of the three forms of Poincaré-invariant quantum mechanics [6] (further PiQM) due to the peculiarities of constructing the interaction operator the light-front dynamics the most used [7–9]. In PiQM calculations the authors use the constituent masses of quarks, the value of which differs from the current masses. Moreover if for the $u\bar{d}$-sector mesons it is often assumed that $m_u = m_d$ [7, 9] then the mass values of s-constituent quark differs [10, 11]. This feature is especially important in the study of the $K^\pm \to \pi^\pm \ell^\mp \nu_{\ell^\mp}$ semileptonic decay since the form-factor $f_+(t)$ determined with Δf accuracy, depended on the difference in quark masses [3]. It follows from the above that further study of semileptonic decays in constituent quark models based on the Poincaré group is an important task of flavour physics.

The paper presents a procedure for calculating the form factors of semileptonic decays in point form PiQM. The main advantage of this form of dynamics is the equality 4-velocities operators with and without interaction. Section 2 describes the procedure for obtaining an integral representation of the semileptonic decay constant in the case of different constituent quark masses. In section 3 showed that for the case of equal masses, the obtained expressions at zero momentum transfer lead to results that coincide with the general theoretical assumptions.

As a result of the work based on the model parameters obtained from $\pi^\pm$-meson leptonic decays using pseudoscalar density constant and calculated relations a study of the pion electromagnetic form-factor for various transferred momenta was carried out. It is shown that the usage of the constituent quark structure leads to results which in good agreement with the experimental data.

2. Semileptonic decay constant calculation

Below we demonstrate the technique for calculating the matrix element between two pseudoscalar mesons based on point form of Poincare-invariant quantum mechanics. In case of pseudoscalar meson semileptonic decay matrix element parameterized by

$$\langle \mathbf{Q}', M' \,|J^\mu|\, \mathbf{Q}, M\rangle = \frac{1}{(2\pi)^3} \frac{f_+(q^2)(Q'+Q)^\mu + f_-(q^2)(Q'-Q)^\mu}{\sqrt{4\ \omega_{M'}(\mathbf{Q}')\ \omega_{M'}(\mathbf{Q})}}, \tag{1}$$

where Q ($Q^2 = M^2$), Q' ($Q'^2 = M'^2$) are 4-momentums of initial and final mesons respectively, $\omega_M(\mathbf{Q}) = \sqrt{\mathbf{Q}^2 + M^2}$ and $q^2 = (Q' - Q)^2$. Since in point form of PiQM 4-velocities with and without interaction are coincide relation (1) write in terms

$$\mathbb{V}^\mu = \frac{Q^\mu}{M}, \quad \mathbb{V}'^\mu = \frac{Q'^\mu}{M'}. \tag{2}$$

After some calculation from (1) and (2) one can obtain

$$\langle \mathbf{Q}', M' \,|J^\mu|\, \mathbf{Q}, M\rangle = \frac{1}{(2\pi)^3} \frac{f_+(q^2)(\mathbb{V}'^\mu\ M' + \mathbb{V}^\mu\ M) + f_-(q^2)(\mathbb{V}'^\mu\ M' - \mathbb{V}^\mu\ M)}{\sqrt{4\ \mathbb{V}_0 M\ \mathbb{V}'_0 M'}}. \tag{3}$$

In proposed model pseudoscalar ($J=\ell=S=\mu=0$) meson state vector in the quark basis $|\mathbf{p}_1,\lambda_1,\mathbf{p}_2,\lambda_2\rangle$ with masses m_q, $m_{\bar{Q}}$ is defined by the following expression:

$$|\mathbf{Q},M\rangle=\sum_{\lambda_1,\lambda_2}\sum_{\nu_1,\nu_2}\int \mathbf{dk}\;\Phi\left(\mathrm{k},\beta_{q\bar{Q}}\right)\sqrt{\frac{\omega_{m_q}(\mathbf{p}_1)\,\omega_{m_{\bar{Q}}}(\mathbf{p}_2)}{\omega_{m_q}(\mathbf{k})\,\omega_{m_{\bar{Q}}}(\mathbf{k})\,\mathbb{V}_0}}\times$$

$$\times\Omega\left(\begin{smallmatrix}0 & 0 & 0\\ \nu_1,&\nu_2,&0\end{smallmatrix}\right)(\theta_{\mathrm{k}},\phi_{\mathrm{k}})\;D^{1/2}_{\lambda_1,\nu_1}(\mathbf{n}_{W_1})\;D^{1/2}_{\lambda_2,\nu_2}(\mathbf{n}_{W_2})\,|\mathbf{p}_1,\lambda_1,\mathbf{p}_2,\lambda_2\rangle\;. \tag{4}$$

In (4) the notation is introduced

$$\Omega\left(\begin{smallmatrix}\ell & S & J\\ \nu_1,&\nu_2,&0\end{smallmatrix}\right)(\theta_{\mathrm{k}},\phi_{\mathrm{k}})=Y_{\ell m}(\theta_{\mathrm{k}},\phi_{\mathrm{k}})\mathrm{C}\left(\begin{smallmatrix}s_1 & s_2 & S\\ \nu_1 & \nu_2 & \lambda\end{smallmatrix}\right)\mathrm{C}\left(\begin{smallmatrix}\ell & S & J\\ m & \lambda & \mu\end{smallmatrix}\right),$$

where $\mathrm{C}\left(\begin{smallmatrix}s_1 & s_2 & S\\ \nu_1 & \nu_2 & \lambda\end{smallmatrix}\right)$, $\mathrm{C}\left(\begin{smallmatrix}\ell & S & J\\ m & \lambda & \mu\end{smallmatrix}\right)$ Clebsch-Gordan coefficients of $SU(2)$ group and $D_{\lambda,\nu}(\mathbf{n}_W)$ Wigner's rotation functions [5]. Meson wave function in (4) is subject of the normalization condition

$$\int_0^\infty \mathrm{dk}\;\mathrm{k}^2\left|\Phi\left(\mathrm{k},\beta_{q\bar{Q}}\right)\right|^2=1, \tag{5}$$

where $\mathbf{k}$ – relative quarks momentum [5, 6].

Substitution state vector (4) in matrix element (1) leads to

$$\langle\mathbf{Q}',M'|J^\mu|\mathbf{Q},M\rangle=\sum_{\lambda_1,\lambda_2}\sum_{\nu_1,\nu_2}\sum_{\lambda_1',\lambda_2'}\sum_{\nu_1',\nu_2'}\int \mathbf{dk}'\;\mathbf{dk}\;\Phi^*\left(\mathrm{k}',\beta_{q'\bar{Q}'}\right)\Phi\left(\mathrm{k},\beta_{q\bar{Q}}\right)\times \tag{6}$$

$$\times\sqrt{\frac{\omega_{m_{q'}}(\mathbf{p}_1')\,\omega_{m_{\bar{Q}'}}(\mathbf{p}_2')}{\omega_{m_{q'}}(\mathbf{k}')\,\omega_{m_{\bar{Q}'}}(\mathbf{k}')\,\mathbb{V}_0'}}\sqrt{\frac{\omega_{m_q}(\mathbf{p}_1)\,\omega_{m_{\bar{Q}}}(\mathbf{p}_2)}{\omega_{m_q}(\mathbf{k})\,\omega_{m_{\bar{Q}}}(\mathbf{k})\,\mathbb{V}_0}}\Omega\left(\begin{smallmatrix}0 & 0 & 0\\ \nu_1',&\nu_2',&0\end{smallmatrix}\right)(\theta_{\mathrm{k}'},\phi_{\mathrm{k}'})\,\Omega\left(\begin{smallmatrix}0 & 0 & 0\\ \nu_1,&\nu_2,&0\end{smallmatrix}\right)(\theta_{\mathrm{k}},\phi_{\mathrm{k}})\times$$

$$\times\langle\mathbf{p}_1',\lambda_1',\mathbf{p}_2',\lambda_2'|J^\mu_{quark}|\,\mathbf{p}_1,\lambda_1,\mathbf{p}_2,\lambda_2\rangle\;D^{*1/2}_{\lambda_1',\nu_1'}(\mathbf{n}'_{W_1})\,D^{*1/2}_{\lambda_2',\nu_2'}(\mathbf{n}'_{W_2})\,D^{1/2}_{\lambda_1,\nu_1}(\mathbf{n}_{W_1})\,D^{1/2}_{\lambda_2,\nu_2}(\mathbf{n}_{W_2}),$$

where J^μ_{quark} – quark operator current [12]

$$J^\mu_{quark}=\bar{\psi}_{q'}\gamma^\mu(1+\varepsilon\gamma_5)\psi_q,\quad \varepsilon=\pm1. \tag{7}$$

After some calculation from (6) and (7) one can obtain

$$\langle\mathbf{Q}',M'|J^\mu|\mathbf{Q},M\rangle=\sum_{\lambda_1,\lambda_2}\sum_{\nu_1,\nu_2}\sum_{\lambda_1',\lambda_2'}\sum_{\nu_1',\nu_2'}\int \mathbf{dk}'\;\mathbf{dk}\;\Phi^*\left(\mathrm{k}',\beta_{q'\bar{Q}'}\right)\Phi\left(\mathrm{k},\beta_{q\bar{Q}}\right)\times \tag{8}$$

$$\times\sqrt{\frac{\omega_{m_{q'}}(\mathbf{p}_1')\,\omega_{m_{\bar{Q}'}}(\mathbf{p}_2')}{\omega_{m_{q'}}(\mathbf{k}')\,\omega_{m_{\bar{Q}'}}(\mathbf{k}')\,\mathbb{V}_0'}}\sqrt{\frac{\omega_{m_q}(\mathbf{p}_1)\,\omega_{m_{\bar{Q}}}(\mathbf{p}_2)}{\omega_{m_q}(\mathbf{k})\,\omega_{m_{\bar{Q}}}(\mathbf{k})\,\mathbb{V}_0}}\Omega\left(\begin{smallmatrix}0 & 0 & 0\\ \nu_1',&\nu_2',&0\end{smallmatrix}\right)(\theta_{\mathrm{k}'},\phi_{\mathrm{k}'})\,\Omega\left(\begin{smallmatrix}0 & 0 & 0\\ \nu_1,&\nu_2,&0\end{smallmatrix}\right)(\theta_{\mathrm{k}},\phi_{\mathrm{k}})\times$$

$$\times\left(e_q\frac{1}{(2\pi)^3}\frac{\bar{u}_{\lambda_1'}(\mathbf{p}_1',m_q')}{\sqrt{2\omega_{m_{q'}}(\mathbf{p}_1')}}\gamma^\mu(1+\varepsilon\gamma_5)\frac{u_{\lambda_1}(\mathbf{p}_1,m_q)}{\sqrt{2\omega_{m_q}(\mathbf{p}_1)}}\langle\mathbf{p}_2',\lambda_2'|\mathbf{p}_2,\lambda_2\rangle+\right.$$

$$\left.+e_{\bar{Q}}\frac{1}{(2\pi)^3}\frac{\bar{v}_{\lambda_2}(\mathbf{p}_2,m_{\bar{Q}})}{\sqrt{2\omega_{m_{\bar{Q}}}(\mathbf{p}_2)}}\gamma^\mu(1+\varepsilon\gamma_5)\frac{v_{\lambda_2'}(\mathbf{p}_2',m_{\bar{Q}'})}{\sqrt{2\omega_{m_{\bar{Q}'}}(\mathbf{p}_2')}}\langle\mathbf{p}_1',\lambda_1'|\mathbf{p}_1,\lambda_1\rangle\right)\times$$

$$\times\;D^{*1/2}_{\lambda_1',\nu_1'}(\mathbf{n}'_{W_1})\,D^{*1/2}_{\lambda_2',\nu_2'}(\mathbf{n}'_{W_2})\,D^{1/2}_{\lambda_1,\nu_1}(\mathbf{n}_{W_1})\,D^{1/2}_{\lambda_2,\nu_2}(\mathbf{n}_{W_2})\,.$$

Following calculation will be carried out in Breit's generalized system [12] where

$$\mathbf{V_Q} + \mathbf{V'_{Q'}} = 0; \tag{9}$$

this feature leads to the following relation for boost operators $B(\mathbf{u})$

$$B(\mathbf{u_{Q'}}) = B(-\mathbf{u_Q}), \quad \mathbf{u_Q} = \frac{\mathbf{Q}}{\omega_M(\mathbf{Q}) + M}. \tag{10}$$

Relations

$$D_{\sigma,\lambda}(\mathbf{n}_W)u_\sigma(\mathbf{p},m) = B(\mathbf{u})u_\lambda(\mathbf{k},m), \quad U(\mathbf{u})|\mathbf{k},\lambda\rangle = \sqrt{\frac{\omega_m(\mathbf{p})}{\omega_m(\mathbf{k})}}\sum_\sigma D_{\sigma,\lambda}(\mathbf{n}_W)|\mathbf{p},\sigma\rangle \tag{11}$$

with considering (10) leads matrix element (8) to

$$\langle \mathbf{Q}', M'\,|J^\mu|\,\mathbf{Q}, M\rangle = \sum_{\nu_1,\nu_2}\sum_{\nu'_1,\nu'_2}\int \mathbf{dk'dk}\ \Phi^*\left(\mathrm{k}',\beta_{q'\bar{Q}'}\right)\Phi\left(\mathrm{k},\beta_{q\bar{Q}}\right)\frac{1}{4\pi\sqrt{\mathbb{V}'_0\mathbb{V}_0}}\delta_{\nu_1,-\nu_2}\delta_{\nu'_1,-\nu'_2}\times \tag{12}$$

$$\times\left(\frac{e_q}{(2\pi)^3}\frac{\bar{u}_{\nu'_1}(\mathbf{k}',m'_q)}{\sqrt{2\omega_{m_{q'}}(\mathbf{k}')}}B^{-1}(\mathbf{u_{Q'}})\gamma^\mu(1+\varepsilon\gamma_5)B(\mathbf{u_Q})\frac{u_{\nu_1}(\mathbf{k},m_q)}{\sqrt{2\omega_{m_q}(\mathbf{k})}}\langle -\mathbf{k}',\nu'_2|U(\boldsymbol{\upsilon}_\mathbf{Q})|-\mathbf{k},\nu_2\rangle + \right.$$

$$\left. +\frac{e_{\bar{Q}}}{(2\pi)^3}\frac{\bar{\upsilon}_{\nu_2}(-\mathbf{k},m_{\bar{Q}})}{\sqrt{2\omega_{m_{\bar{Q}}}(\mathbf{k})}}B^{-1}(\mathbf{u_Q})\gamma^\mu(1+\varepsilon\gamma_5)B(\mathbf{u_{Q'}})\frac{\upsilon_{\nu'_2}(-\mathbf{k}',m_{\bar{Q}'})}{\sqrt{2\omega_{m_{\bar{Q}'}}(\mathbf{k}')}}\langle \mathbf{k}',\nu'_1|U(\boldsymbol{\upsilon}_\mathbf{Q})|\mathbf{k},\nu_1\rangle\right),$$

where also was used feature of the generalized Breit system $U^\dagger(\mathbf{u_{Q'}})U(\mathbf{u_Q}) = U(\boldsymbol{\upsilon}_\mathbf{Q})$, $\boldsymbol{\upsilon}_\mathbf{Q} = \mathbf{Q}/\omega_m(\mathbf{Q})$ and

$$\Omega\left(\begin{smallmatrix}0 & 0 & 0\\ \nu_1, & \nu_2, & 0\end{smallmatrix}\right)(\theta_\mathrm{k},\phi_\mathrm{k}) = \frac{1}{2\sqrt{\pi}}\delta_{\nu_1,-\nu_2}, \quad \Omega\left(\begin{smallmatrix}0 & 0 & 0\\ \nu'_1, & \nu'_2, & 0\end{smallmatrix}\right)(\theta_{\mathrm{k}'},\phi_{\mathrm{k}'}) = \frac{1}{2\sqrt{\pi}}\delta_{\nu'_1,-\nu'_2}. \tag{13}$$

Final simplification of the matrix element we will achieve using momentums

$$\mathbf{k}_{1,2} = \mathbf{k} \pm \boldsymbol{\upsilon}_\mathbf{Q}\left((\varpi+1)\omega_m(\mathbf{k}) - \sqrt{\varpi^2-1}\,\mathrm{k}\,\cos\theta_\mathrm{k}\right), \quad \varpi = (\mathbb{V}\cdot\mathbb{V}'), \tag{14}$$

which were obtained by the transformation of the quark state-vectors by $U(\boldsymbol{\upsilon}_\mathbf{Q})$. Integration over $\mathbf{k}'$ leads to integral representation of the semileptonic decay matrix element in point form of PiQM:

$$\langle \mathbf{Q}', M'\,|J^\mu|\,\mathbf{Q}, M\rangle = \frac{1}{(2\pi)^3}\sum_{\nu_1,\nu_2}\sum_{\nu'_1,\nu'_2}\int \mathbf{dk}\ \Phi\left(\mathrm{k},\beta_{q\bar{Q}}\right)\frac{1}{4\pi\sqrt{\mathbb{V}'_0\mathbb{V}_0}}\delta_{\nu_1,-\nu_2}\delta_{\nu'_1,-\nu'_2}\times \tag{15}$$

$$\times\left(e_q\frac{\bar{u}_{\nu'_1}(\mathbf{k}_2,m'_q)}{\sqrt{2\omega_{m_{q'}}(\mathbf{k}_2)}}B(\boldsymbol{\upsilon}_\mathbf{Q})\gamma^\mu(1+\varepsilon\gamma_5)\frac{u_{\nu_1}(\mathbf{k},m_q)}{\sqrt{2\omega_{m_q}(\mathbf{k})}}\Phi^*\left(\mathrm{k}_2,\beta_{q'\bar{Q}'}\right)\sqrt{\frac{\omega_{m_{\bar{Q}}}(\mathbf{k}_2)}{\omega_{m_{\bar{Q}}}(\mathbf{k})}}D_{\nu'_2,\nu_2}(\mathbf{n}_{W_2}) + \right.$$

$$\left. +e_{\bar{Q}}\frac{\bar{\upsilon}_{\nu_2}(-\mathbf{k},m_{\bar{Q}})}{\sqrt{2\omega_{m_{\bar{Q}}}(\mathbf{k})}}B(-\boldsymbol{\upsilon}_\mathbf{Q})\gamma^\mu(1+\varepsilon\gamma_5)\frac{\upsilon_{\nu'_2}(-\mathbf{k}_1,m_{\bar{Q}'})}{\sqrt{2\omega_{m_{\bar{Q}'}}(\mathbf{k}_1)}}\Phi^*\left(\mathrm{k}_1,\beta_{q'\bar{Q}'}\right)\sqrt{\frac{\omega_{m_q}(\mathbf{k}_1)}{\omega_{m_q}(\mathbf{k})}}D_{\nu'_1,\nu_1}(\mathbf{n}_{W_1})\right),$$

where

$$B(\boldsymbol{\upsilon}_\mathbf{Q}) = \frac{1-(\boldsymbol{\upsilon}_\mathbf{Q}\cdot\boldsymbol{\gamma})\gamma_0}{\sqrt{1-\boldsymbol{\upsilon}^2_\mathbf{Q}}}. \tag{16}$$

Following spinor part calculation will be conducted for studied pion semileptonic decay case.

3. $\pi^{\pm} \to \pi^0 \ell^{\pm} \nu_{\ell^{\pm}}$ decay in point form of PiQM

Parametrization of pion semileptonic decay is given by [7] (see relation (1))

$$ {}_{\pi^0}\langle \mathbf{Q}', M' | J^{\mu} | \mathbf{Q}, M \rangle_{\pi^+} = \frac{\sqrt{2}}{(2\pi)^3} \frac{f_+(q^2)(Q'+Q)^{\mu} + f_-(q^2)(Q'-Q)^{\mu}}{\sqrt{4\ \omega_{M'}(\mathbf{Q}')\ \omega_{M'}(\mathbf{Q})}}. \quad (17) $$

Since pion mass difference $M - M' \simeq 0$ relation (17) in terms of 4-velocities could be written as

$$ {}_{\pi^0}\langle \mathbf{Q}', M' | J^{\mu} | \mathbf{Q}, M \rangle_{\pi^+} = \frac{\sqrt{2}}{(2\pi)^3} \frac{f_{\pi^+}(q^2)(\mathrm{V}'^{\mu}\ M' + \mathrm{V}^{\mu}\ M)}{\sqrt{4\ \mathrm{V}_0 M\ \mathrm{V}_0' M'}}. \quad (18) $$

For Breit's generalized system $\mathbf{V_Q} = \{0, 0, \mathrm{V_Q}\}$, $\mathbf{V}'_{\mathbf{Q}'} = \{0, 0, -\mathrm{V_Q}\}$ (see (9)) right side of equation (18) takes form

$$ {}_{\pi^0}\langle \mathbf{Q}', M' | J^{\mu} | \mathbf{Q}, M \rangle_{\pi^+} = \frac{\sqrt{2}}{(2\pi)^3} f_{\pi^+}(q^2)\ \frac{K^{\mu}}{\sqrt{\mathrm{V}_0' \mathrm{V}_0}}, \quad (19) $$

where

$$ K^{\mu} = \frac{1}{2\sqrt{2}\sqrt{M'M}} \{(M+M')\sqrt{1+\varpi}, 0, 0, (M-M')\sqrt{-1+\varpi}\}. \quad (20) $$

From (20) it can be seen that for case $M - M' \simeq 0$ with small transfer momentum (i.e. $\varpi \to 1$) third component of quark current can be neglected. The remaining zero component of the quark current will be investigated below in the presented model.

In what follows we will assume that the values of the masses of constituent u- and d-quarks are equal [7–11]. In that case from relations (15) and (19) one can obtain integral representation of pion form-factor:

$$ f_{\pi^+}(q^2) = \frac{1}{4\pi} \sum_{\nu_1,\nu_2} \sum_{\nu_1',\nu_2'} \int \mathbf{dk}\ \Phi\left(\mathrm{k}, \beta_{u\bar{d}}\right) \frac{1}{2\,\omega_{m_q}(\mathbf{k})} \frac{1}{K^0} \delta_{\nu_1,-\nu_2} \delta_{\nu_1',-\nu_2'} \times \quad (21) $$

$$ \times \left(e_u\, \bar{u}_{\nu_1'}(\mathbf{k}_2, m_q) B(\boldsymbol{v}_{\mathbf{Q}}) \gamma^0 (1+\varepsilon\gamma_5) u_{\nu_1}(\mathbf{k}, m_q) \Phi^*\left(\mathrm{k}_2, \beta_{d\bar{d}}\right) D_{\nu_2',\nu_2}(\mathbf{n}_{W_2}) + \right. $$

$$ \left. + e_{\bar{d}}\, \bar{v}_{\nu_2}(-\mathbf{k}, m_q) B(-\boldsymbol{v}_{\mathbf{Q}}) \gamma^0 (1+\varepsilon\gamma_5) v_{\nu_2'}(-\mathbf{k}_1, m_q) \Phi^*\left(\mathrm{k}_1, \beta_{u\bar{u}}\right) D_{\nu_1',\nu_1}(\mathbf{n}_{W_1}) \right). $$

In relation (21) and further will be used notation $m_u = m_d = m_q$. Spinor part calculation leads to rather cumbersome answer, but in should be notes, that the part proportional to the matrix γ_5 equal to zero [7]. Using the specified result expression (21) can be written as

$$ f_{\pi^+}(q^2) = \int \mathbf{dk}\ \frac{1}{K^0}\ \Phi\left(\mathrm{k}, \beta_{u\bar{d}}\right) \left(e_u\ j_{em.}(\mathrm{k}, m_q) \Phi^*\left(\mathrm{k}_2, \beta_{d\bar{d}}\right) + e_{\bar{d}}\ j_{em.}(\mathrm{k}, m_q) \Phi^*\left(\mathrm{k}_1, \beta_{u\bar{u}}\right)\right), \quad (22) $$

where $j_{em.}$ – zero component of electromagnetic current

$$ j_{em.}(\mathrm{k}, m_q) = \frac{1}{4\pi} \frac{1}{2\,\omega_{m_q}(\mathbf{k})} \sum_{\nu_1,\nu_1'} \bar{u}_{\nu_1'}(\mathbf{k}_2, m_q) B(\boldsymbol{v}_{\mathbf{Q}}) \gamma^0 u_{\nu_1}(\mathbf{k}, m_q) D_{-\nu_1',-\nu_1}(\mathbf{n}_{W_2}) = \quad (23) $$

$$= \frac{1}{4\pi}\,\frac{1}{2\,\omega_{m_q}(\mathbf{k})}\,\sum_{\nu_1,\nu_1'} \bar{\upsilon}_{-\nu_1}(-\mathbf{k},m_q)B(-\boldsymbol{\upsilon}_{\mathbf{Q}})\gamma^0\upsilon_{-\nu_1'}(-\mathbf{k}_1,m_q)D_{\nu_1',\nu_1}(\mathbf{n}_{W_1}).$$

Evaluation of expression (23) for zero transfer momentum (i.e. $\varpi = 1$) using property of point form of PiQM $M \equiv M_0(\mathbf{k}) = 2\omega_{m_q}(\mathbf{k})$ [5, 6] with assumption $\beta_{u\bar{d}} = \beta_{u\bar{u}} = \beta_{d\bar{d}}$ one can obtain

$$f_{\pi^+}(0) = \frac{1}{4\pi}\int \mathbf{dk}\,|\Phi\,(\mathrm{k},\beta_{u\bar{d}})|^2\,(e_u + e_{\bar{d}})\,. \tag{24}$$

Solid angle integration taking into account normalization condition of wave function (5) leads to value of semileptonic decay constant $f_{\pi^+}(0) = e_u + e_{\bar{d}} = 1$, that taking into account the parametrization of $\pi^{\pm} \to \pi^0\ell^{\pm}\nu_{\ell^{\pm}}$ (see (18)) semileptonic decay leads to well-known result [13].

Below we well study the pion form-factor for various transferred momentum.

4. Electromagnetic pion form-factor investigation in point form of PiQM

Since expression (22) corresponds to quark electromagnetic current in this section we will test proposed model for electromagnetic pion form-factor investigation in case of different transfer momentum.

Determination of the developed model parameters will be conducted using the integral representation of the pseudoscalar ($a_P = -1$) meson decay constant [14]

$$f_I(m_q,m_{\bar{Q}},\beta^I_{q\bar{Q}}) = \sqrt{\frac{3}{2}\frac{1}{\pi}}\int \mathrm{dk}\;\mathrm{k}^2\;\Phi(\mathrm{k},\beta^I_{q\bar{Q}})\times \tag{25}$$

$$\times\sqrt{\frac{W^+_{m_q}(\mathrm{k})\;W^+_{m_{\bar{Q}}}(\mathbf{k})}{M_0(\mathbf{k})\;\omega_{m_q}(\mathbf{k})\;\omega_{m_{\bar{Q}}}(\mathbf{k})}}\left(1 + a_P\frac{|\mathbf{k}|^2}{W^+_{m_q}(\mathbf{k})W^+_{m_{\bar{Q}}}(\mathbf{k})}\right),$$

where

$$W^{\pm}_m(\mathrm{k}) = \omega_m(\mathbf{k}) \pm m.$$

In our approach determination the values of the constituent masses of quarks will be conducted using the pseudoscalar density constant g_P, which is determined from the expression

$$\langle 0\,|J_5|\,\mathbf{Q},M\rangle = -i\frac{1}{(2\pi)^{3/2}}\frac{g_P}{\sqrt{2}\;\omega_{M_P}(\mathbf{Q})}. \tag{26}$$

After similar calculations (see [14]) one can obtain an integral representation of the pseudoscalar density constant in point form PiQM:

$$g_P(m_q,m_{\bar{Q}},\beta^P_{q\bar{Q}}) = \sqrt{\frac{3}{2}\frac{1}{\pi}}\int \mathrm{dk}\;\mathrm{k}^2\frac{\Phi(\mathrm{k},\beta^P_{q\bar{Q}})}{\sqrt{\omega_{m_q}(\mathbf{k})\;\omega_{m_{\bar{Q}}}(\mathbf{k})}}\times \tag{27}$$

$$\times\sqrt{M_0(\mathbf{k})}\left(\sqrt{W^+_{m_q}(\mathbf{k})W^+_{m_{\bar{Q}}}(\mathbf{k})} + \sqrt{W^-_{m_q}(\mathbf{k})W^-_{m_{\bar{Q}}}(\mathbf{k})}\right).$$

Calculation of constituent masses of quarks and parameters of wave functions will be carried out by solving the system of equations [15]

$$\begin{cases} \frac{1}{2}\left(\hat{m}_u + \hat{m}_d\right) = (3.45 \pm 0.4)\ \text{MeV}, \\ f_P(m_u, m_d, \beta^P_{u\bar{d}}) = f_P^{(\text{exp.})}, \\ \left(\hat{m}_u + \hat{m}_d\right)\ g_P(m_u, m_d, \beta^P_{u\bar{d}}) = f_P^{(\text{exp.})} M_P^2, \end{cases} \tag{28}$$

where $f_P^{(\text{exp.})}$, M_P experimental values of the decay constant $\pi^{\pm}$–meson and its mass, $\hat{m}_q$ quark current mass [4]. Since in our approach masses of constituent u– and d–quarks are equal the solution to the system (28) with oscillator wave function

$$\Phi(\text{k}, \beta_{q\bar{Q}}) = \frac{2}{\pi^{1/4}\left(\beta^I_{q\bar{Q}}\right)^{3/2}} \exp\left[-\frac{\text{k}^2}{2\left(\beta_{q\bar{Q}}\right)^2}\right]. \tag{29}$$

leads to the following values of the basic parameters of the model:

$$m_{u,d} = 221.52 \pm 2.82\ \text{MeV},\ \beta^P_{u\bar{u}} = \beta^P_{d\bar{d}} = \beta^P_{u\bar{d}} = 374.51 \pm 4.28\ \text{MeV}. \tag{30}$$

Using the obtained model parameters and integral representation of pion decay constant calculation (see (22)) we will study the form-factor f_{π^+} for different transfer momentum (see fig. 1).

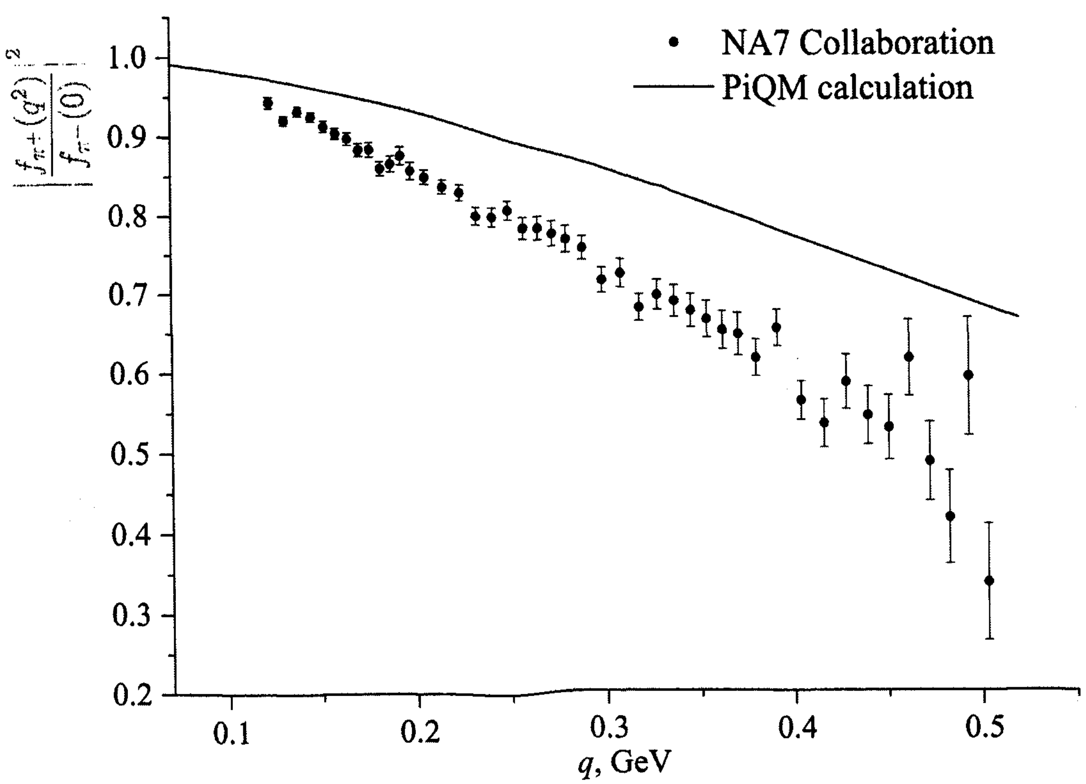

Figure 1: Behaviour of the pion electromagnetic form-factor

Experimental data taken from [16].

For generality let's assume the quark structure. In that case relation (22) will be modified by [17]

$$F_q(q^2 = t) = \frac{e_q}{1 - \frac{a}{6\, m_q^2}\, q^2}, \quad a = 0.3\,. \tag{31}$$

After similar calculation one can obtain following behaviour (see fig. 2).

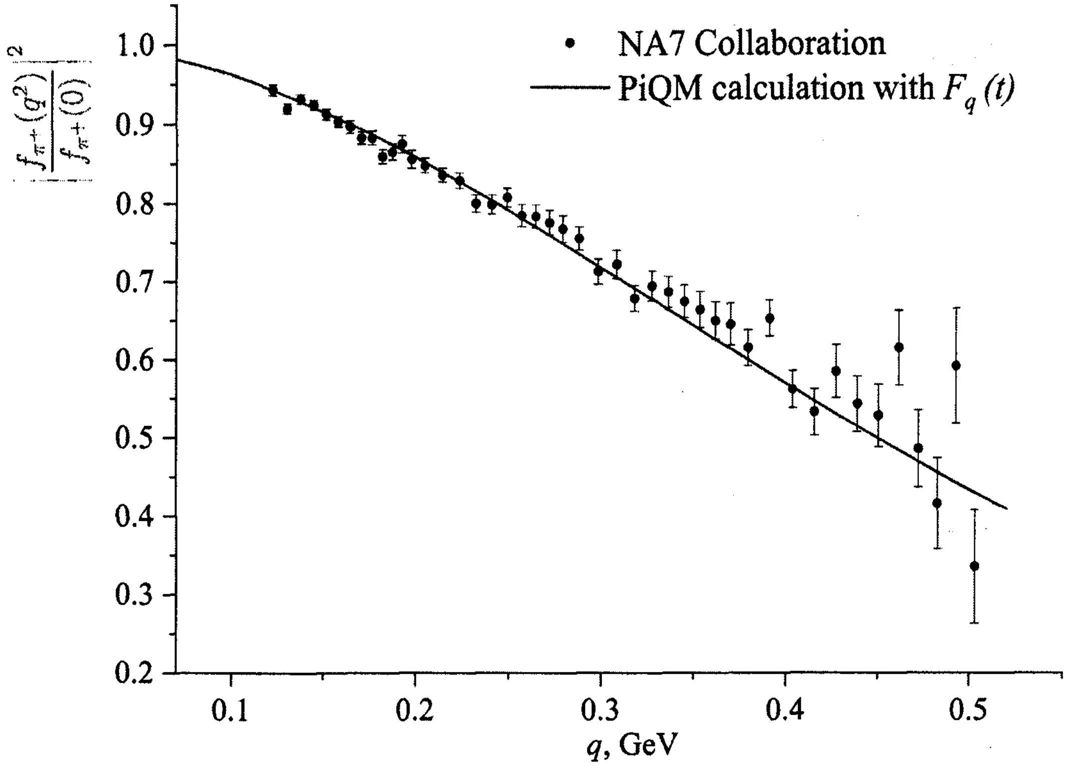

Figure 2: Behaviour of the pion electromagnetic form-factor taking into account quark structure

Conclusion and remarks

The work is dedicated to the study of semileptonic decays of hadrons in a compound quark model based on the point form of Poincaré-invariant quantum mechanics. In the course of the work, the authors developed a procedure for calculating the form -factors of pseudoscalar mesons of $h \to h'\ell^{\pm}\nu_{\ell^{\pm}}$ decay of the type. As a test of the obtained relations the decay $\pi^{\pm} \to \pi^0\ell^{\pm}\nu_{\ell_{\pm}}$ was studied: it is shown that at zero momentum transfer the model calculations coincide with the general theoretical expressions.

The authors note that the work is purely methodical: previously developed calculation $V(P) \to P(V)\gamma$ scheme is generalized to the case of studying of the semileptonic decay form-factor. The calculation scheme developed in this work can be used for investigation $K^{\pm}$, $D^{\pm}$, $B^{\pm}$-mesons decay as well as for studying the physical properties of quarkonia.

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