Nonlinear Dynamics and Applications. Vol. 28 (2022) 92 - 103

Leptonic Decay Widths for the Composite System of Two Spin Particles with Equal Masses in the Relativistic Quasipotential Approach

Yu. D. Chernichenko*
International Center for Advanced Studies,
Gomel State Technical University, Belarus

L. P. Kaptari†
Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, Dubna, Russia

The new relativistic WKB expression for leptonic decay widths of the relativistic systems of two fermions with equal masses interacting by means of the funnel-type potentials is obtained. Quasipotential equation is solved by the WKB approximation. The behavior of the relativistic leptonic decay widths of vector mesons was investigated. Comparison of the behavior for new expression with its nonrelativistic spinless and relativistic spin and spinless analogues are given. Consideration is conducted within the framework of completely covariant quasipotential approach in the Hamiltonian formulation of quantum field theory, via a transition to the relativistic configurational representation in the case of two relativistic spin particles of equal masses.

PACS numbers: 11.55.Hx, 13.60.Hb, 11.55.Fv, 13.85.Lg, 13.85.Qk

Keywords: relativistic quasipotential approach, relativistic configurational representation, leptonic decay widths, quantum chromodynamics

1. Introduction

In the two-particle approximation the square of the module of the Bethe-Salpeter (BS) wave function of two charged particles, $\chi_{\rm BS}(x)$, at $x = (x_0, \mathbf{r}) = 0$ and hence at the relative time $x_0 = 0$ is connected with the leptonic decay widths of 1^- states, $\Gamma_{n,\ell=0}(e^+e^-)$, as [1] (see also Refs. [2–5])

$$\Gamma_{n,\ell=0}(e^+e^-) = 16\pi\alpha^2 e_q^2 \frac{|\chi_{\rm BS}(x=0)|^2}{M_n^2}|_{\ell=0}, \tag{1}$$

where α is the fine-structure constant, e_q is the quark charge in the units of e with the number of its colours $N_c = 3$, and M_n is the total c.i.s. energy for given level n of the composite system of two relativistic particles ($q\bar{q}$ or e^+e^- system) with equal masses $m_1 = m_2 = m$.

*E-mail: `chyud@mail.ru`; E-mail: `chern@gstu.by`
†E-mail: `post@jinr.ru`

Presentation of the square of the BS wave function in zero, $|\chi_{\rm BS}(x=0)|^2$, through square of the wave function in zero for the $\ell = 0$ state, $|\psi_0^{\rm nr}(0)|^2$, corresponding to the nonrelativistic Schrödinger equation in the case of two particles of equal masses with the potential

$$V(r) = V_{\rm conf}(r) - \frac{\alpha_s}{r}, \tag{2}$$

where $V_{\rm conf}(r)$ is the confining potential ($V_{\rm conf}(0) = 0$), and α_s is the strong coupling constant, was executed in Refs.[6, 7]. Their result as at $E_n > 0$, so and at $E_n < 0$ is ($\hbar = c = 1$)

$$|\psi_0^{\rm nr}(0)|^2 = F(v_n^{\rm nr})\frac{m^2}{4\pi^2}v_n^{\rm nr}\frac{dE_n}{dn}, \tag{3}$$

where $v_n^{\rm nr} = \sqrt{E_n/m}$ is the nonrelativistic velocity of free quark with mass m and kinetic energy $E_n/2$ for given level n, and the Coulomb S-factor $F(v)$ is determined by expression

$$F(v) = \frac{\pi\alpha_s}{v}\left[1 - \exp\left(-\frac{\pi\alpha_s}{v}\right)\right]^{-1}, \tag{4}$$

moreover, for $\alpha_s = 0$ ($F(v) = 0$) the nonrelativistic expression (3) moves over to expression, obtained in Refs. [8, 9].

Generalization of the nonrelativistic expression (3) was executed in Ref. [1]. Their method is based on replacement of the full BS interaction kernel by an appropriate instantaneous $q\bar{q}$ interaction (the Salpeter approximation). In terms of a first approximation for the Salpeter wave function, $\Psi_0^{\rm rel}(r)$ ($\ell = 0$), they obtained ($V_{\rm conf}(0) = 0, \hbar = c = 1$)

$$|\Psi_0^{\rm rel}(0)|^2 = F(v_n)\frac{M_n^2}{16\pi^2}v_n\frac{dM_n}{dn},\ M_n \lessgtr 2m, \tag{5}$$

where

$$v = \sqrt{1 - \frac{4m^2}{M^2}} \tag{6}$$

is the relativistic velocity of a free quark with mass m and the total energy $M/2 = (2m + E)/2$, moreover, for $\alpha_s = 0$ ($F(v) = 0$) the relativistic Eq. (5) moves over to expression, obtained earlier in Refs. [10, 11][1)].

Other approach for findings leptonic decay widths of vector mesons is founded on using of the relativistic quasipotential (RQP) approach [13] in quantum field theory. In the RQP approach [14] via a transition to the relativistic three-dimensional $\mathbf{r}$ representation [15] in the case of two relativistic spin particles with equal masses m, the BS wave function in zero, $\chi_{\rm BS}(x=0)$, can be expressed through the RQP wave functions in the Lobachevsky momentum space, $\Psi_{M_Q}(\mathbf{k})$, with a relative three-momentum $\mathbf{k}$ and in the $\mathbf{r}$ representation, $\psi_{M_Q}(\mathbf{r})$, as

$$\chi_{\rm BS}(x=0) = \frac{1}{(2\pi)^3}\int d\Omega_{\mathbf{k}}\Psi_{M_Q}(\mathbf{k}) = \lim_{r\to i\lambda}\psi_{M_Q}(\mathbf{r}), \tag{7}$$

where $d\Omega_{\mathbf{k}} = mc^2 d\mathbf{k}/E_k$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid $E_q^2 - c^2\mathbf{q}^2 = m^2c^4$, $\lambda = \hbar/mc$ is the Compton wavelengths of the relativistic particle with mass m, a relative three-momentum $\mathbf{q}$

1) The existence of such a relation for the relativistic case was earlier offered but is not proved in Ref. [12].

and the energy $E_q = q_0 = c\sqrt{m^2c^2 + \mathbf{q}^2}$, that is connected with the total c.i.s. energy $\sqrt{s}$ by expression $M_Q = \sqrt{s} = 2c\sqrt{m^2c^2 + \mathbf{q}^2} = 2E_q$ [15]. Thereby, Eq. (7) provides the correct relationship between BS function and RQP wave functions in the Lobachevsky momentum space and in the relativistic three-dimensional $\mathbf{r}$ representation.

The relativistic modification of the formula (1) in RQP approach [14] via a transition to the relativistic $\mathbf{r}$ representation [15] for the case of interaction between two relativistic spinless particles with equal masses m by potential $V(r) = -\alpha_s/r + \sigma r^s, \sigma, s > 0$, it was performed in Ref.[16]. Their result is

$$\Gamma_{n,\ell=0}(e^+e^-) = \frac{16\pi\alpha^2 e_q^2}{M_n^2} \left|\psi_{M_Q}(0)\right|^2 \Big|_{\ell=0} = \tag{8}$$

$$= \alpha^2 e_q^2 \frac{m\tilde{\alpha}_s}{\pi\hbar^3 M_n^2} \frac{dM_n}{dn} \left| F\left(1 - \frac{\tilde{\alpha}_s}{2\sin\kappa_n}, 1; 2; 1 - e^{-2i\kappa_n}\right) \right|^2,$$

where $\tilde{\alpha}_s = \alpha_s/\hbar c$, $\kappa_n = \arccos(M_n/2mc^2)$, and $F(a, b; c; z)$ is the hypergeometric function.

We shall note also Refs. [17] in which the RQP approach [14] via a transition to the relativistic $\mathbf{r}$ representation [15] for the system of two relativistic fermions with equal masses m, that interacts by means of the singular confining potential (2), is used to calculate the leptonic decay widths for the vector and pseudoscalar mesons in the semiclassical (WKB) approximation. Moreover, the Coulomb part of potential (2) that is used in Refs. [17], i.e.

$$V_{\mathrm{Coul}} = -\frac{\alpha_s}{r}, \tag{9}$$

is the image of scalar part of one boson exchange potential and its possesses the QCD-like behaviour in the Lobachevsky momentum space [18]. Their result for the vector mesons with relative orbital moment $\ell \geq 0$ can be presented through the hypergeometric function in the form

$$\Gamma_{n,\ell}(e^+e^-) = \frac{16\pi\alpha^2 e_q^2}{M_n^2} \left|\psi_{M_Q}(0)\right|^2 = \frac{2\alpha^2 e_q^2 \Gamma^4(\ell+1)\left(2\sinh\chi_n\right)^{2\ell+1} e^{-2\chi_n(\ell+1)}}{\pi\lambda^3 mc^2 \Gamma^2(2\ell+2) M_n^2} \times \tag{10}$$

$$\times L_{\mathrm{RQP}}(\chi_n) \left| F\left(\ell + 1 - \frac{i\tilde{\alpha}_s}{2\sinh\chi_n}, \ell+1; 2\ell+2; 1 - e^{-2\chi_n}\right) \right|^2 \frac{dM_n}{dn},$$

where $\Gamma(z)$ is gamma-function,

$$L_{\mathrm{RQP}}(\chi) = \prod_{n=1}^{\ell} \left[1 + \left(\frac{\tilde{\alpha}_s}{2n\sinh\chi}\right)^2\right] S_{\mathrm{RQP}}(\chi), \tag{11}$$

$$S_{\mathrm{RQP}}(\chi) = \frac{X_{\mathrm{RQP}}(\chi)}{1 - \exp[-X_{\mathrm{RQP}}(\chi)]}, \; X_{\mathrm{RQP}}(\chi) = \frac{\pi\tilde{\alpha}_s}{\sinh\chi}, \tag{12}$$

are the relativistic spinless resummations Coulomb L and S factors as functions of the rapidity $\chi = \mathrm{arcosh}(M/2mc^2)$, which appears in considered RQP approach[2)] (see [19–21]), moreover, $\sinh\chi = v/\sqrt{1-v^2}$, and the expression (10) for $\ell = 0$ and $\chi_n = i\kappa_n = i\arccos(M_n/2mc^2)$ moves over to expression (8).

[2)] We shall remind that the relativistic spinless resummations Coulomb L and S factors in (11) and (12) have the correct relativistic and ultrarelativistic limits unlike the relativistic S factors, presented in Refs. [22–24] (for details, see Refs. [19–21]).

In RQP approach [14] the relativistic WKB expression for the leptonic decay widths of vector mesons with energy M_n and with relative orbital moment $\ell \geq 0$ for the case of two relativistic spinless particles with arbitrary masses m_1, m_2, that interacts by means of potential (2), has, taking into account the formulas (1) and (7), following the form [25]

$$\Gamma_{n,\ell}(e^+e^-) = \frac{4\alpha^2 e_q^2 \mu \Gamma^2(\ell+1)}{\pi \lambda'^3 m'^2 c^2 \Gamma^2(2\ell+2) M_n^2} \left(2\sinh\chi'_n\right)^{2\ell+1} L_{\rm RQP}\left(\chi'_n\right) \frac{dM_n}{dn}. \tag{13}$$

Here $\mu = m_1 m_2/(m_1+m_2)$ is the usual reduced mass, $\lambda' = \hbar/m'c$ is the Compton wavelengths of the effective relativistic particle with mass $m' = \sqrt{m_1 m_2}$, a relative three-momentum $\mathbf{q}'$ and the energy $E_{q'} = c\sqrt{m'^2c^2 + \mathbf{q}'^2}$, emerging instead of the system of two particles and carrying the total c.i.s. energy $\sqrt{s}$ of the interacting particles proportional to the energy $E_{q'}$ (see [26, 27]): $M = \sqrt{s} = c\sqrt{m_1^2c^2+\mathbf{q}^2} + c\sqrt{m_2^2c^2+\mathbf{q}^2} = (m'/\mu)E_{q'}$, and the relativistic spinless resummations Coulomb L and S factors as functions of the rapidity χ', which appears in considered RQP approach [19–21], are given in Eqs. (11) and (12), where now $\chi \to \chi' = \operatorname{arcosh}(\mu M/m'^2c^2)$, $\tilde{\alpha}_s \to \tilde{\alpha}'_s = 2\mu\alpha_s/\hbar m'c$.

In this paper within the framework of completely covariant of the quasipotential RQP approach in Hamiltonian formulation of quantum field theory for the case of two relativistic spin particles with equal masses m [14], we obtain the expression for the leptonic decay widths of vector mesons with relative orbital moment $\ell = 0$. For this we generalize the relativistic modified WKB method proposed in [16] on the case of two relativistic spin particles (quarks) with equal masses, that interacts by means of potential (2).

2. The solutions of RQP equation in WKB approximation

The basis of our consideration is completely covariant RQP equation into the $\mathbf{r}$ representation in terms finite differences constructed in [28] for the RQP wave function $\psi_{M_Q}(\mathbf{r})$ of two relativistic spin particles with equal masses m. For spherically symmetric potential this equation has the form

$$\frac{1}{2mc^2}\left(M_Q - \widehat{H}_0\right)\psi_{M_Q}(\mathbf{r}) = V(\mathbf{r})\hat{A}\left(\frac{\widehat{H}_0}{2mc^2}\right)\psi_{M_Q}(\mathbf{r}). \tag{14}$$

Here $M_Q^2 = Q^2 = (q_1+q_2)^2$, the operator

$$\widehat{H}_0 = 2mc^2\left[\cosh\left(i\lambda\frac{\partial}{\partial r}\right) + \frac{i\lambda}{r}\sinh\left(i\lambda\frac{\partial}{\partial r}\right) - \frac{\lambda^2}{2r^2}\Delta_{\theta,\varphi}\exp\left(i\lambda\frac{\partial}{\partial r}\right)\right] \tag{15}$$

is the operator of free Hamiltonian while $\Delta_{\theta,\varphi}$ is its the angular part, potential $V(\mathbf{r})$ is local in the sense of Lobachevsky geometry and for simplicity depends not from energy M_Q, but the operator $\hat{A}$ is defined by expression

$$\hat{A}\left(\frac{\widehat{H}_0}{2mc^2}\right) = \frac{1}{4}\left[a\left(\frac{\widehat{H}_0}{2mc^2}\right)^2 + b\right], \tag{16}$$

where the spin parameters a and b for vector meson has the following importance

$$a = \frac{1}{2},\ b = \frac{1}{4}. \tag{17}$$

By using the expansion of the RQP wave function $\psi_{M_Q}(\mathbf{r})$ on a Legendre function $P_\mu(z)$ of the first kind,

$$\psi_{M_Q}(\mathbf{r}) = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell \frac{\varphi_\ell(r,\chi)}{r} P_\ell\left(\frac{\mathbf{\Delta}_{q,m\lambda_Q}\cdot \mathbf{r}}{\Delta^0_{q,m\lambda_Q} r}\right),$$

Eq. (14) transformed to the form

$$\left[\widehat{H}_0^{\rm rad} - \cosh\chi + V(r)\hat{A}\left(\widehat{H}_0^{\rm rad}\right)\right]\varphi_\ell(r,\chi) = 0, \qquad (18)$$

where

$$\widehat{H}_0^{\rm rad} = \cosh\left(i\lambda\frac{d}{dr}\right) + \frac{\lambda^2\ell(\ell+1)}{2r(r+i\lambda)}\exp\left(i\lambda\frac{d}{dr}\right)$$

is the radial part of operator free Hamiltonian (15), operator $\hat{A}$ is determined in (16), and χ is the rapidity, which is connected to the 3-momentum $\mathbf{\Delta}_{q,m\lambda_Q}$ and the energy M_Q as[3)]

$$\mathbf{\Delta}_{q,m\lambda_Q} = mc\sinh\chi\mathbf{n}_{\Delta_{q,m\lambda_Q}},\ |\mathbf{n}_{\Delta_{q,m\lambda_Q}}| = 1,\ M_Q = 2\Delta^0_{q,m\lambda_Q},\ \Delta^0_{q,m\lambda_Q} = mc^2\cosh\chi.$$

We will seek the WKB solution of RQP Eq. (18) in the usual form [16, 17, 25]

$$\varphi_\ell(r,\chi) = \exp\left[\frac{i}{\hbar}g(r)\right],\ g(r) = g_0(r) + \frac{\hbar}{i}g_1(r) + \left(\frac{\hbar}{i}\right)^2 g_2(r) + \dots. \qquad (19)$$

With two first terms of the expansion in (19) the WKB solutions with the left, r_L, and the right, r_R, of the classical turning points in the inner region $r_L \le r \le r_R$ has then the form

$$\varphi_\ell^{L,R}(r,\chi) = \frac{C_{L,R}(\chi)}{2\sqrt[4]{[\mathcal{X}^2(r) - R^2(r)][1 + aV(r)X(r)]}}\left\{\exp\left[i\alpha_+^{L,R}(r) \mp \frac{i\pi}{4}\right] + \right. \qquad (20)$$
$$\left. + \exp\left[i\alpha_-^{L,R}(r) \pm \frac{i\pi}{4}\right]\right\},$$

where

$$\alpha_\pm^{L,R}(r) = \frac{1}{\lambda}\int_{r_{L,R}}^{r} dr'\chi_\pm(r'),\ \chi_\pm(r) = \ln\left[\mathcal{X}(r) \pm \sqrt{\mathcal{X}^2(r) - R^2(r)}\right], \qquad (21)$$

$$\mathcal{X}(r) = \frac{2X(r)}{1+\sqrt{1+aV(r)X(r)}},\ X(r) = \cosh\chi - \frac{b}{4}V(r),\ R(r) = \sqrt{1+\frac{\lambda^2\Lambda^2}{r^2}},\ \Lambda = \ell + 1/2,$$

$C_{L,R}$ are the normalization constants, and the turning points, $r_{L,R}$, are branch points of root in the WKB solutions (21) that lead to the condition

$$\mathcal{X}(r_{L,R}) = R(r_{L,R}). \qquad (22)$$

---

3) We shall remind that here all the 4-momentums belong to the upper sheet of the mass hyperboloid $\Delta^2_{q,m\lambda_Q} = \Delta^{02}_{q,m\lambda_Q} - c^2\mathbf{\Delta}^2_{q,m\lambda_Q} = m^2c^4$, where $\lambda_Q = (\lambda^0_Q; \boldsymbol{\lambda}_Q) = Q/\sqrt{Q^2}$ is the 4-velocity of a composite particle with the 4-momentum $Q = q_1 + q_2$, and $\Delta^0_{q,m\lambda_Q}$, $\mathbf{\Delta}_{q,m\lambda_Q}$ are, respectively, the time and spatial components of the 4-vector $\Lambda^{-1}_{\lambda_Q}q = \Delta_{q,m\lambda_Q}$ from the Lobachevsky space (for details, see Ref. [28]).

Condition of applicability of the relativistic WKB solutions (20) has the form

$$\lambda \left| \frac{\cosh \chi_{\text{eff}}(r)}{\chi_+(r) \sinh \chi_{\text{eff}}(r)} \frac{d\chi_+(r)}{dr} \right| \ll 1, \tag{23}$$

where

$$\chi_{\text{eff}}(r) = \text{arcosh}\mathcal{X}_{\text{eff}}(r) = \ln \left(\mathcal{X}_{\text{eff}}(r) + \sqrt{\mathcal{X}^2_{\text{eff}}(r) - 1} \right), \; \mathcal{X}_{\text{eff}}(r) = \cosh \chi_{\text{eff}}(r) = \frac{\mathcal{X}(r)}{R(r)}.$$

In the case of $\ell = 0$, the condition (23) is converted in inequality

$$\lambda \left| \frac{\cosh \chi_S(r)}{\chi_S(r) \sinh \chi_S(r)} \frac{d\chi_S(r)}{dr} \right| \ll 1, \tag{24}$$

where

$$\chi_S(r) = \text{arcosh}\mathcal{X}(\text{r}) = \ln \left[\mathcal{X}(\text{r}) + \sqrt{\mathcal{X}^2(\text{r}) - 1} \right] \tag{25}$$

is the rapidity of relativistic particle of the mass m that moves in the field of potential $V(r)$.

We note that at $a = 0, b = 2/mc^2$ the Eqs. (20)–(25) has coincide with the analogous expressions obtained in the spinless case for arbitrary masses in [25] and taken at $m_1 = m_2 = m$.

3. The relativistic leptonic decay widths in WKB approximation

In the RQP approach the leptonic decay widths of the vector mesons for the state with energy $M_n = 2mc^2 \cosh \chi_n$ for given of level n and of relative orbital moment $\ell = 0$, in accordance with expressions (1), (7), and of WKB solution with the left, r_L, classical turning point in (20), we define by formula

$$\Gamma_{n,\ell=0}(e^+e^-) = \frac{16\pi\alpha^2 e_q^2}{M_n^2} \lim_{r \to i\lambda} \left| e^{-\pi\tilde{\rho}/2} \Gamma(1 + i\tilde{\rho}) \frac{\varphi_0^L(r, \chi_n)}{r} \right|^2, \tag{26}$$

where

$$\tilde{\rho} = \frac{\tilde{\alpha}_s a \cosh \chi}{4}, \; \tilde{\alpha}_s = \frac{\alpha_s}{\lambda}.$$

Thus, we see that, in the spinor case, the function

$$\psi_0(r, \chi) = e^{-\pi\tilde{\rho}/2} \Gamma(1 + i\tilde{\rho}) \varphi_0^L(r, \chi)$$

is the physical wave function of s-wave state of the composite system that consistes of two relativistic spinor particles of equal masses, interacting by means of potential (2).

For potential (2) the WKB approximation of the RQP radial wave function $\varphi_\ell^L(r, \chi_n)$ in region $r \in (r_L; r_R)$ in accordance with Eqs. (20) and (21) can be presented in the form

$$\varphi_\ell^L(r, \chi_n) = \frac{C_\ell(\chi_n)}{\sqrt[4]{[\mathcal{X}^2(r) - R^2(r)][1 + aV(r)X(r)]}} \sin \left\{ \frac{1}{\lambda} \int_{r_L}^{r} dr' \left[\chi_+(r') - \ln R(r') \right] + \frac{\pi}{4} \right\}, \tag{27}$$

where the normalization constant $C_\ell(\chi_n)$ is found from the normalization condition

$$4\pi \int_0^\infty dr \left|\varphi_\ell^L(r,\chi_n)\right|^2 = 1, \; \ell \geq 0. \tag{28}$$

In the region of applicability of the relativistic WKB method the argument of sine in (27) is the quickly oscillating function. Therefore, the square of sine in (28) can be replaced, either as in the nonrelativistic case, on its the average importance equal 1/2 [25]. Instead of Eq. (28) we then obtain

$$2\pi|C_\ell(\chi_n)|^2 \int_{r_L}^{r_R} \frac{dr}{\sqrt{[\mathcal{X}^2(r) - R^2(r)][1 + aV(r)X(r)]}} = 1. \tag{29}$$

Differentiation of the WKB quantization condition of energy levels [29]

$$\int_{r_L}^{r_R} dr \left[\chi_+(r) - \ln R(r)\right] = \pi\lambda\left(n + \frac{1}{2}\right), \; n = 0, 1, \ldots, \; \ell \geq 0,$$

on the total energy $M_n = 2mc^2 \cosh \chi_n$ in accordance with the expressions (21) and (22), gives condition

$$\int_{r_L}^{r_R} \frac{dr}{\sqrt{[\mathcal{X}^2(r) - R^2(r)][1 + aV(r)X(r)]}} = 2\pi\lambda mc^2 \frac{dn}{dM_n}. \tag{30}$$

From Eqs. (29) and (30) we find

$$|C_\ell(\chi_n)|^2 = \frac{1}{4\pi^2\lambda mc^2} \frac{dM_n}{dn}. \tag{31}$$

Then, either as in the Refs. [6, 25], the WKB radial wave function (27) of the potential (2) at $\ell \geq 0$ for enough of the large value of $\rho = r/\lambda, r \in (r_L; r_R)$, but such, where the Coulomb interaction (9) will dominate in the potential (2), can be approximated by the Coulomb radial s-wave function for which its the exact form is the known [28, 30, 31]

$$\begin{aligned} \varphi_0^{\text{Coul}}(\rho,\chi) = 2\pi C_0^{\text{Coul}}(\chi) e^{iB\chi - \chi + i(\rho - \tilde{\rho})\chi}(\rho - \tilde{\rho}) \times \\ \times F\left(1 - iB, 1 - i(\rho - \tilde{\rho}); 2; 1 - e^{-2\chi}\right). \end{aligned} \tag{32}$$

Here $C_0^{\text{Coul}}(\chi)$ is the normalization constant, the parameter B is defined as

$$B = \frac{\tilde{\alpha}_s(a\cosh^2\chi + b)}{4\sinh\chi} \tag{33}$$

and at $\chi = i\kappa_n$ it is connected with the quantization condition of the energy levels by expression [28, 30]

$$\frac{\tilde{\alpha}_s(a\cos^2\kappa_n + b)}{4\sin\kappa_n} = n, \; n = 1, 2, \ldots, \; 0 < \kappa_n < \pi/2.$$

Comparing of asymptotic expression for the Coulomb wave function in (32),

$$\varphi_0^{\text{Coul}}(\rho,\chi_n)\big|_{\rho\gg1} \sim \frac{2\pi C_0^{\text{Coul}}(\chi_n)e^{-\pi B/2}}{\sinh\chi_n|\Gamma(1-iB)|}\sin\left[\rho\chi_n+\delta_0^{\text{Coul}}(\chi_n)\right],$$

with the asymptotic form of WKB solution in (27), taken at $\ell=0$,

$$\varphi_0^L(\rho,\chi_n)\big|_{\rho\gg1} \sim \frac{C_0(\chi_n)}{\sqrt{\sinh\chi_n}}\sin\left[\rho\chi_n+\delta_0^{\text{Coul,WKB}}(\chi_n)\right],$$

gives the relationship between the normalization constants

$$\left|2\pi C_0^{\text{Coul}}(\chi_n)\right|^2 = \sinh\chi_n e^{\pi B}|\Gamma(1-iB)|^2\,|C_0(\chi_n)|^2, \tag{34}$$

where

$$\delta_0^{\text{Coul}}(\chi) = B\ln(2\rho\sinh\chi)-\tilde{\rho}\chi+\arg\Gamma(1-iB)$$

is the phase of the Coulomb wave function in (32), and

$$\delta_0^{\text{Coul,WKB}}(\chi) = B\ln(2\rho\sinh\chi)-\tilde{\rho}\chi-B\ln B$$

is its expression in the WKB approximation [29].

Finally, taking into consideration Eqs. (26), (31), (32), and (34), we get expression for the relativistic leptonic decay widths of the vector mesons in s-wave state and with energy M_n for the case of the potential (2):

$$\Gamma_{n,\ell=0}(e^+e^-) = \frac{4\alpha^2e_q^2\sinh\chi_n}{\pi\lambda^3mc^2M_n^2}S_{\text{RQP},S}(\chi_n)\frac{dM_n}{dn}. \tag{35}$$

Here

$$S_{\text{RQP},S}(\chi) = \frac{X_{\text{RQP},S}(\chi)}{1-\exp\left[-X_{\text{RQP},S}(\chi)\right]}e^{-\pi\tilde{\rho}}\left|\Gamma(2+i\tilde{\rho})F(1+iB,-i\tilde{\rho};2;1-e^{-2\chi})\right|^2 \tag{36}$$

is the resummation Coulomb S factor for the system of two relativistic spin particles with equal masses m, that interacts by means of the potential (9), which appears in considered RQP approach[4)], where the value $X_{\text{RQP},S}(\chi)$ is connected with parameter B in (33) by expression

$$X_{\text{RQP},S}(\chi) = 2\pi B = \frac{\pi\tilde{\alpha}_s(a\cosh^2\chi+b)}{2\sinh\chi},$$

which can be expressed through the relativistic velocity (6) as

$$X_{\text{RQP},S}(v) = \frac{\pi\tilde{\alpha}_s(a+b-bv^2)}{2v\sqrt{1-v^2}},\ \sinh\chi = \frac{v}{\sqrt{1-v^2}}. \tag{37}$$

We note that at $a=0, b=2/mc^2$ Eq. (35) moves over to Eq. (13) for the case spinless, taken at $\ell=0, m_1=m_2=m$.

---

4) We shall remind that the S factor in (36) reproduces both the known nonrelativistic limit in the case spinless ($a=0, b=2/mc^2$) and the expected relativistic ($v\to1$) and ultrarelativistic ($m\to0$) limits for the importance of parameters a and b in (17) (for details, see Refs.[30–33]).

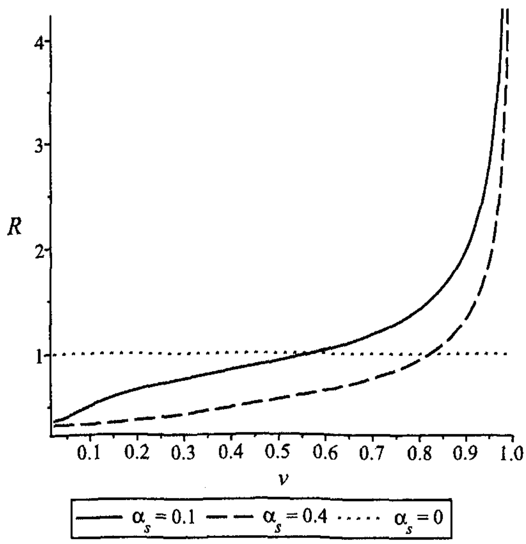

Figure 1. Ratio $R = R(v)$ of the relativistic Eq. (35) for the leptonic decay widths of vector mesons (in the system of units where $\hbar = c = 1$) at the importance of spin parameters vector mesons $a = 1/2$ and $b = 1/4$ in (17) for the case of the funnel-like potential (2) at importances of the coupling constant of (solid curve) $\alpha_s = 0.1$, (dashed curve) $\alpha_s = 0.4$, and (dotted line) $\alpha_s = 0$ to its nonrelativistic spinless analog given by Eqs. (1), (3), and (4).

On Figs. 1–3, we present the behavior of the functions $R = R(v)$ that are defined as the ratio of the relativistic leptonic decay widths of s-state vector mesons, which is presented by Eqs. (35), (36) and (37), taken at $\hbar = c = 1$, to its the nonrelativistic spinless and relativistic spin and spinless of the analogues, presented by expressions (1), (3), (4) and (10), (12), taken at $\ell = 0, \hbar = c = 1$, and (12), (13), taken at $\ell = 0, m_1 = m_2 = m, \hbar = c = 1$, accordingly. The curves on Figs. 1–3 are built for two importances of the coupling constant $\alpha_s = 0.1$ (the solid curves) and $\alpha_s = 0.4$ (the dashed curves), which answers the importance of spin parameters vector mesons $a = 1/2$ and $b = 1/4$ in (17). The dotted line corresponds to the case of $\alpha_s = 0$. From Figs. 1–3 we can see that in the nonrelativistic region of the velocity values of v ($v \ll 1$) the spin of quarks, which form the spin parameters $a = 1/2$ and $b = 1/4$ of vector mesons, affects substantially the behavior of the functions $R = R(v)$ and, hence, on the behavior of the leptonic decay widths of vector mesons. This effect is associated with the influence of the importance of spin parameters vector mesons ($a = 1/2$ and $b = 1/4$) on the behavior of the threshold S factor (36) in the region of small values of the velocity v. This is the so-called Sommerfeld effect [34, 35]. This influence of the spin parameters on the behavior of the threshold S factor in the region of small values of the velocity v was studied in detail in Refs. [30–33].

When the velocity v grows, the influence of the spin parameters $a = 1/2$ and $b = 1/4$ of vector mesons on the behavior of function $R = R(v)$ becomes weaker, and $R \to 1$ in the relativistic limit ($v \to 1$), as one can see from Fig. 3, that is, this the influence disappears. An unbounded growth of the function $R = R(v)$ in the relativistic limit ($v \to 1$) on Figs. 1 and 2, is connected with the violation of correlation (7), which establishes a correct relation between the Bethe–Salpeter function at $x = 0$ and the RQP wave function, calculated in the three-dimensional relativistic $\mathbf{r}$ representation at $r = i\lambda$, rather than at $r = 0$.

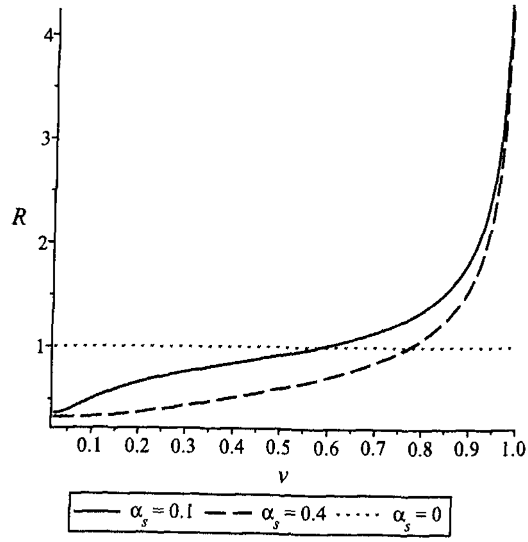

Figure 2. Too, as on Fig. 1, but to its the relativistic spin analogue, which is given by expression (10), taken at $\ell = 0, \hbar = c = 1$.

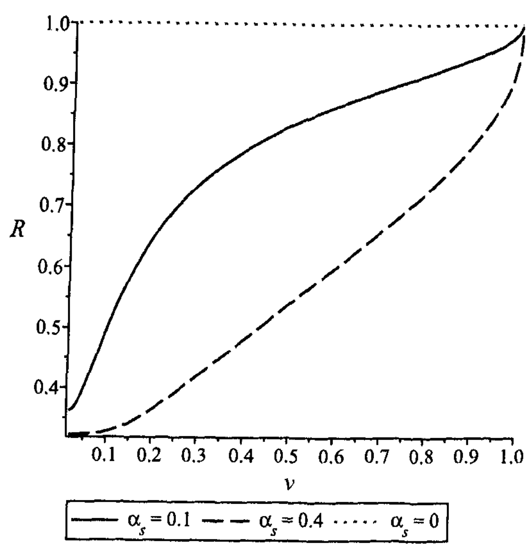

Figure 3. Too, as on Fig. 1, but to its the relativistic spinless analogue, which is given by expression (13), taken at $m_1 = m_2 = m, \hbar = c = 1$.

Also, Figs. 1–3 show that the behavior of the function $R = R(v)$ depends on the coupling constant α_s in broad region of importances of the velocity v, and this dependence is weak in the nonrelativistic and relativistic regions of importances of the velocity v. Besides, the curves representing the function $R = R(v)$ on Fig. 3 for the coupling constant importances of $\alpha_s = 0.1$ and $\alpha_s = 0.4$ tend to unity in the relativistic limits ($v \to 1$), since the relativistic threshold S factors in (12) and (36) have a correct relativistic limit equal to unity [19–21, 30–33].

We emphasize, that the influence of the mass difference between particles (quarks), which forms the composite system, it is necessary take into account, when we investigate the behavior of the threshold S factor (see [32, 33]) and, hence, it influences on the behavior of the leptonic decay widths of vector mesons in the s-wave state.

4. Conclusion

In the present study, the new relativistic expression for the leptonic decay widths of vector mesons in s-wave state have been obtained on the basis of the RQP approach in the relativistic semiclassical approximation. The present analysis has been performed for the case where relativistic spin quarks of equal mass m that form vector mesons interact via a funnel-like potential including a purely confining part, which is not singular, and a singular part in the form of a Coulomb-like chromodynamical potential. For this aim the fully covariant finite-difference RQP equation in the three-dimensional relativistic $\mathbf{r}$ representation [15] for the case of interaction between two relativistic spin particles of equal mass has been solved by the relativistic WKB method. The condition of applicability of the WKB approximation has been established. The comparison of the new expression with its the relativistic spinless and relativistic spin and spinless of analogs is executed. The influence of the quark spin, which form the spin parameters $a = 1/2$ and $b = 1/4$ of vector mesons, on the behavior of leptonic decay widths of vector mesons in the s-wave state has been explored. This the influence of the quark spin on the behavior of leptonic decay widths of vector mesons in the s-wave state is essential in the region of small values of the velocity v (it the so-called Sommerfeld effect), but when the velocity v grows, its influence becomes weaker, and in the relativistic limit ($v \to 1$) its the influence disappears. The influence of variations in the coupling constant on the behavior of the leptonic decay widths of vector mesons has been revealed. It has been shown that, at $a = 0$ and $b = 2/mc^2$, the new expression for the leptonic decay widths of vector mesons reduces to its relativistic spinless analog.

Since the new relativistic expression for the relativistic semiclassical leptonic decay widths of vector mesons has been obtained within a fully covariant method and has a correct connection with the Bethe–Salpeter function, one can expect that this expression takes into account more adequately both the relativistic character of interacting particles and their spin.

Acknowledgments

I am grateful to O.P. Solovtsova for a discussion on the results of this study, enlightening comments, and technical support. Stimulating conversations with Yu.A. Kurochkin, V.V. Andreev, and A.V. Kiselev in the course of the present investigation, as well a their comments and discussion on its results, are also gratefully acknowledged.

This work was supported in part by the International Program of Cooperation between the Republic of Belarus and Joint Institute for Nuclear Research, Dubna, and by the Convergence-2025 Research Program (for the period spanning the years 2021 and 2025) of the Republic of Belarus (Microscopic World, Plasma, and Universe Subprogram).

References

[1] B. Durand and L. Durand, Phys. Rev. D, **30**, 1904 (1984).
[2] E. Etim and L. Schülke, Nuovo Cim. A **77**, 347 (1983).
[3] V. A. Matveev, B. V. Struminsky, A. N. Tavkhelidze, Preprint JINR P-2524, Dubna, 1965.
[4] W. F. Weisskopf, R. Van Royen, Nuovo Cim., **50**, 617 (1967); **51**, 583 (1967).
[5] R. Barbieri, R. Gatto, R. Kögerler and Z. Kunszt, Phys. Lett. B, **57**, 455 (1975).
[6] J. S. Bell and P. Pasupathy, Z. Phys. C, **2**, 183 (1979).
[7] N. Fröman and P. O. Fröman, J. Phys. (Paris), **42**, 1491 (1981).
[8] C. Quigg and J. Rosner, Phys. Rev. D, **17**, 2364 (1978).
[9] J. S. Bell and P. Pasupathy, Phys. Lett. B, **83**, 389 (1979).
[10] B. Durand and L. Durand, Phys. Rev. D, **25**, 2312 (1982).
[11] B. Durand and L. Durand, Phys. Lett. B, **113**, 338 (1982).
[12] E. A. Tainov, Phys. Lett. B, **97**, 283 (1980); Z. Phys. C, **10**, 87 (1981).
[13] A. A. Logunov and A. N. Tavkhelidze, Nuovo Cim., **29**, 380 (1963).
[14] V. G. Kadyshevsky, Nucl. Phys. B, **6**, 125 (1968).
[15] V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, Nuovo Cim. A, **55**, 233 (1968).
[16] N. B. Skachkov, I. L. Solovtsov, Sov. J. Nucl. Phys. **31**, 686 (1980).
[17] A. V. Sidorov, N. B. Skachkov, Theor. Math. Phys., **46**, 141 (1981); Preprint JINR P2-80-45, Dubna, 1980; V. I. Savrin, A. V. Sidorov, and N. B. Skachkov, Hadronic J. **4**, 1642 (1981).
[18] V. I. Savrin and N. B. Skachkov, Lett. Nuovo Cimento **29**, 363 (1980).
[19] O. P. Solovtsova, Yu. D. Chernichenko, Phys. Atom. Nucl. **73**, 1612 (2010).
[20] O. P. Solovtsova, Yu. D. Chernichenko, Theor. Math. Phys. **166**, 194 (2011).
[21] Yu. D. Chernichenko, Vestsi Nats. Akad. Navuk Belarusi, Ser. fiz.-mat. navuk, **4**, 81 (2009).
[22] A. H. Hoang, Phys. Rev. D **56**, 7276 (1997).
[23] J.-H. Yoon and Ch.-Y. Wong, Phys. Rev. C **61**, 044905 (2000); J. Phys. G: Nucl. Part. Phys. **31**, 149 (2005).
[24] A. B. Arbuzov, Nuovo Cimento A **107**, 1263 (1994).
[25] Yu. D. Chernichenko, V. V. Kondratjuk, Phys. At. Nucl. **81**, 51 (2018).
[26] V. G. Kadyshevsky, M. D. Mateev, and R. M. Mir-Kasimov, Yad. Fiz., **11**, 692 (1970).
[27] V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, Sov. J. Purt. Nucl., **2**, 69 (1972).
[28] Yu. D. Chernichenko, Phys. At. Nucl. **80**, 707 (2017).
[29] Yu. D. Chernichenko, Phys. At. Nucl. **83**, 488 (2020).
[30] Yu. D. Chernichenko, Phys. At. Nucl. **82**, 158 (2019).
[31] Yu. D. Chernichenko, O. P. Solovtsova, and L. P. Kaptar, in *Proceedings of the XXVII Anniversary Seminar "Nonlinear Phenomena in Complex Systems", NPCS'2020, May 19–22, 2020, Minsk, Belarus*, Nonlinear Dynamics and Applications, Minsk, 2020, Vol. 26, p. 39.
[32] Yu. D. Chernichenko, O. P. Solovtsova, and L. P. Kaptari, Nonlin. Phen. Compl. Syst. **23**, 449 (2020).
[33] Yu. D. Chernichenko, Phys. At. Nucl. **84**, 339 (2021).
[34] A. Sommerfeld, *Atombau und Spektrallinien* (Vieweg, Braunschweig, 1939), Vol. 2.
[35] G. Gamov, Z. Phys. **51**, 204 (1928).