

Testing perturbative QCD via the Bjorken sum rule

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We study the polarized Bjorken sum rule at low momentum transfers in the range $0.22 < Q < 1.73$ GeV with the four-loop $N^3\text{LO}$ expression for the coefficient function $C_{\text{Bj}}(\alpha_s)$ in the framework of the common QCD perturbation theory. The analysis of the C_{Bj} -series gives a hint to its asymptotic nature manifesting itself in the region $Q < 1$ GeV. This leaves no room for higher twist correction when approaching the data most closely.

PACS numbers: 13.60.Hb

Keywords: deep inelastic lepton-hadron scattering, Bjorken sum rule, perturbative theory

1. Introduction

Higher-twist parameters are important ingredients of the nucleon spin structure. Their extraction from experimental studies is relatively complicated as they are most pronounced at low momentum transfer Q , where a theoretical analysis includes both the perturbative and nonperturbative components related to each other. As a consequence, the uncertainty caused by the inevitable truncation of the perturbation series affects the reliability of information about the higher-twist terms. The analysis of deep inelastic scattering data by provides us with a test site for combining both the perturbative and non-perturbative contributions in the low energy domain. In particular, the Bjorken Sum Rule (BSR) [1] is a renown target ground for testing different possibilities [2, 3]. Fortunately, recent Jefferson Lab data [4] give information on the spin-dependent BSR amplitude $\Gamma_1^{p-n}(Q^2)$ behavior close to the confinement/hadronization scale. The BSR amplitude $\Gamma_1^{p-n}(Q^2)$ is given by a sum of two series in powers of $1/Q^2$ (OPE higher twists corrections) and in powers of the QCD running coupling $\alpha_s(Q^2)$ (pQCD radiative corrections). Until very recently, the pQCD contribution to BSR was known [5] up to the third order $\sim \alpha_s^3$. So far, the corresponding expression has been used in many studies, in particular, for extraction of the α_s values [6] and the higher twists (HT) terms [7] at low momentum scales.

The four-loop expression for the pQCD contribution to the BSR became recently available in Ref. [8]. It gives us a reasonable motivation for a new extended QCD analysis of the combined JLab data on $\Gamma_1^{p-n}(Q^2)$ at low $0.05 < Q^2 < 3.0$ GeV² accounting for up to α_s^4 -order. We found that the inclusion of higher-order (HO)) of PT terms demonstrates a "duality" between HO and HT, in the sense that HT terms are absorbed, with good numerical accuracy, into HO terms. As a result, HT coefficients decrease. At the same time, we observed that the description of the data is improved only up to the two-loop order of PT, which may be a signal of the asymptotic character of PT series in the region close to the Landau pole.

2. The perturbative QCD contribution

The BSR claims that the difference of the proton and neutron structure functions integrated over all possible values

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 [g_1^p(x, Q^2) - g_1^n(x, Q^2)] dx, \quad (1)$$

of the Bjorken variable x in the limit of large four-momentum squared of the exchanged virtual photon, $Q^2 \rightarrow \infty$, is equal to $g_A/6$, with $g_A = 1.267 \pm 0.004$ [9], the nucleon axial charge defined from the neutron β -decay data.

Commonly, one represents the Bjorken integral (1) as a sum of the perturbative and the higher twist contributions

$$\Gamma_1^{p-n}(Q^2) = \frac{g_A}{6} \left[1 - \Delta_{\text{Bj}}(Q^2) \right] + \sum_{i=2}^{\infty} \frac{\mu_{2i}}{Q^{2i-2}}. \quad (2)$$

In our notation the perturbative QCD Δ_{Bj} correction is defined by the coefficient function $C_{\text{Bj}}(\alpha_s)$: $\Delta_{\text{Bj}} = 1 - C_{\text{Bj}}(\alpha_s)$. The perturbative correction has a form of the power series in the QCD running coupling $\alpha_s(Q^2)$. At the up-to-date four-loop (N^3LO) level in the massless case it looks like

$$\Delta_{\text{Bj}}^{\text{PT}}(Q^2) = \sum_{k \leq 4} c_k \alpha_s^k(Q^2). \quad (3)$$

Here, the numerical expansion coefficients c_i in the modified minimal subtraction ($\overline{\text{MS}}$) scheme, for three active flavors, $n_f = 3$, read $c_1 = 1/\pi = 0.31831$, $c_2 = 0.36307$ [10], $c_3 = 0.65197$ [5] and $c_4 = 1.8042$ [8]. Besides, the four-loop running coupling $\alpha_s(Q^2)$ is defined as a solution of the Renormalization Group (RG) equation

$$\frac{d\alpha_s}{dL} = \beta(\alpha_s); \quad \beta(\alpha_s) = \sum_{0 \leq k \leq 3} \beta_k \alpha_s^{k+2}, \quad (4)$$

where $L = \ln(\mu^2/\Lambda^2)$ and β_k are the coefficients of the β -function. For our purposes, it is convenient to represent the β -function in the form

$$\beta(\alpha_s) = -\beta_0 \alpha_s^2 (1 + b_1 \alpha_s + b_2 \alpha_s^2 + b_3 \alpha_s^3 + \dots), \quad (5)$$

with $b_i = \beta_i/\beta_0$, the ratios of the β -function coefficients. For three flavors the coefficients are $\beta_0 = 9/4\pi = 0.7162$, $b_1 = 0.5659$, $b_2^{\overline{\text{MS}}} = 0.4530$ [11] and $b_3^{\overline{\text{MS}}} = 0.6770$ [12]. In the current analysis we use the exact solutions of the RG equation (4) in the $\overline{\text{MS}}$ -scheme at the scale $\mu = Q$.

2.1. The Q^2 -dependence

Let us analyze the Q^2 -dependence of the BSR in the PT approach in different orders (NLO, N^2LO and N^3LO) of the perturbative expansion (3). As a normalization point, we use the most accurate α_s -value at $Q = M_Z$, $\alpha_s(M_Z) = 0.1184 \pm 0.0007$ [9, 13]. In order to take into account flavor thresholds, we apply the matching conditions for the values of $\alpha_s(Q^2)$ which are rather nontrivial in higher PT orders (see Refs. [14–16]). Following to analysis in Ref. [17], our matched calculation for the four-loop $\overline{\text{MS}}$ -coupling gives $\Lambda^{(n_f=3)} = 336 \pm 10$ MeV. Note, we obtain practically the same results, but with larger errors, if we choose the pseudo-observable value $R(M_Z^2) = 1.03904 \pm 0.00087$ as a normalization point [18], which leads to the four-loop running coupling equal to $\alpha_s(M_Z) = 0.1190 \pm 0.0026$.

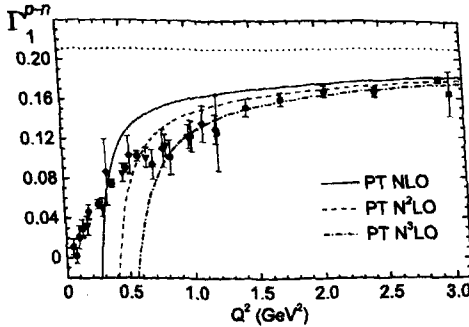


FIG. 1: Perturbative part of the BSR as a function of the momentum transfer squared Q^2 in different orders of standard PT approach against the combined set of the Jefferson Lab [4, 20] and SLAC [19] data.

In Fig. 1, we illustrate the behavior of the perturbative part of the BSR in different orders in α_s . For completeness, we also show here the combined SLAC and JLab data on $\Gamma_1^{p-n}(Q^2)$ used in our analysis. The SLAC data points [19] are denoted by squares, the JLab CLAS Hall A 2002 data – by downward pointing triangles, the JLab CLAS Hall B 2003 data – by diamonds [20], and the most recent JLab data [4] – by circles. The horizontal dotted line represents the limiting value $\Gamma_1^{p-n}(Q^2 \rightarrow \infty) = g_A/6$.

One can see that at $Q^2 \geq 0.7 \text{ GeV}^2$ the four-loop approximation describes the data quite well. Moreover, the corresponding curve passes close to the central values of several data points, although the experimental accuracy (which is of the same order as both the three- and four-loop contributions) does not allow one to make a definite choice between four- and three-loop approximations.

At the same time, at $Q^2 \leq 0.7 \text{ GeV}^2$ the four-loop approximation describes the data equally bad as the three- and two-loop ones. This is a signal of the necessity to account for HT contributions, and it will be strongly dependent on the order of PT used for its extraction [7].

This situation may be considered as an indication of the transition of PT series to the asymptotic regime for $Q^2 \sim 0.7 \text{ GeV}^2$. Let us explore this possibility in more detail.

2.2. Convergence of the PT expansions

Clearly, at low Q^2 a value of the strong coupling is quite large, questioning the convergence of perturbative QCD series. The PT power series truncated after four-loop order (c.f. Eq. (3)) reads

$$\Delta_{\text{BJ}}^{\text{PT}}(\alpha_s) = 0.3183 \alpha_s + 0.3631 \alpha_s^2 + 0.6520 \alpha_s^3 + 1.804 \alpha_s^4 = \sum_{i \leq 4} \delta_i(\alpha_s),$$

where δ_i is the i -th term. The qualitative resemblance of the coefficients pattern to the factorial growth did not escape our attention although the more definite statements, if possible, would require much more efforts. This observation allows one to estimate the value of $\alpha_s \sim 1/3$ providing a similar magnitude of three- and four-loop contributions to the BSR.

To test that, we present in Fig. 2 the relative contributions of separate terms in the four-loop expansion (6).

One can see that the four-loop PT correction becomes equal to the three-loop one at $Q^2 = 2 \text{ GeV}^2$ and noticeably overshooting it (note that the slopes of these contributions are quite

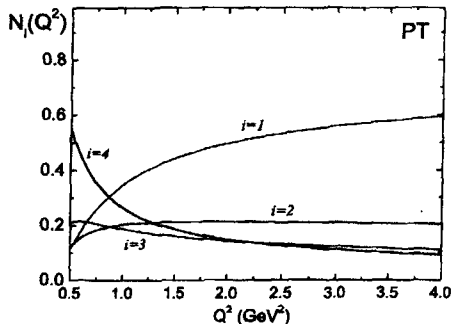


FIG. 2: The Q^2 -dependence of the relative contributions at the four-loop level in the PT approach. Four-loop PT order overshoots the three-loop one at $Q^2 \leq 2 \text{ GeV}^2$, so it does not improve the accuracy of the PT series compared to the three-loop one.

close in the relatively wide Q^2 region) for $Q^2 \sim 1 \text{ GeV}^2$ which may be considered as an extra argument supporting an asymptotic character of the PT series in this region.

3. Higher twists contribution

3.1. The results of the fit

Now, using expression (2) fitted to the above mentioned experimental data [4, 20] we extract the coefficients μ_{2i} of the higher twist OPE corrections. The minimal borders of fitting domains in Q^2 are settled from the *ad hoc* restriction $\chi^2 < 1$ and monotonous behavior of the resulting fitted curves.

Previously, a detailed higher-twist analysis of the two- and three-loop expansions in powers of α_s was performed in Refs. [7]. Now, we extend the analysis up to an order $\sim \alpha_s^4$.

In Figs. 3 and 4 we present the results of 1- and 3-parametric fits in various orders of PT. The corresponding fit results for higher twist terms, extracted in different orders of PT, are given in Table 1 (all numerical results are normalized to the corresponding powers of the nucleon mass M). From these figures and Table 1 one can see the lower border of analysis Q_{min}^2 shifts up to higher Q^2 scales when increasing the order of PT expansion. This is caused by extra unphysical singularities in the higher-loop strong coupling.

3.2. Sensitivity of the higher twists to Λ_{QCD} variations

In the above analysis, we normalized α_s at the Z -boson mass scale and then fixed the value of the Λ parameter separately in each order in α_s approximation (it was sufficient for understanding the role of the fourth order in the PT perturbative series). However, the corresponding values of the Λ parameter extracted in this way may be different from ones obtained in the direct QCD analysis of the experimental data on the moments of the structure functions (see, e.g., Ref. [21]). Having this in mind, we investigate additionally the sensitivity of the extracted values of the higher twist term μ_4 to the QCD scale parameter Λ in various orders of PT.

In Fig. 5 we show values of the coefficient μ_4 extracted from the JLab data using two-, three- and four-loop PT at $Q_{min}^2 = 0.66 \text{ GeV}^2$ vs the parameter Λ . One can see that the PT does

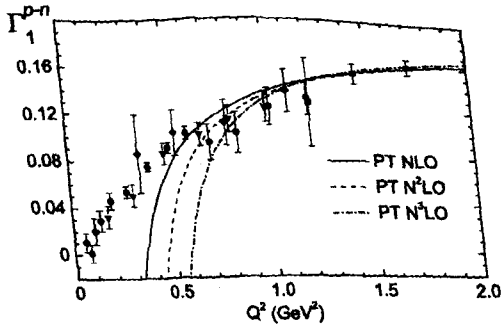


FIG. 3: The one-parametric μ_4 -fits of the BSR JLab data in various (NLO, N^2 LO, N^3 LO) orders of the PT. The four-loop result does not improve the data description compared to the three-loop one.

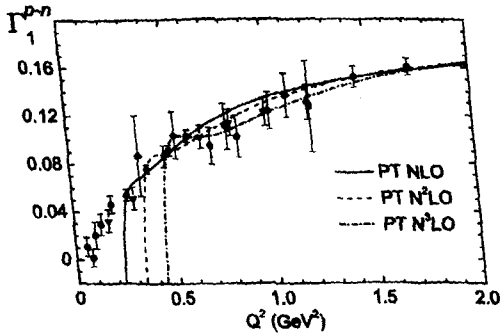


FIG. 4: The three-parametric $\mu_{4,6,8}$ -fits of the BSR JLab data in various (NLO, N^2 LO, N^3 LO) orders of the PT and the all-order APT expansions.

Table 1: Results of higher twist extraction from the JLab data on BSR in various (NLO, N^2 LO, N^3 LO) orders of PT and all orders of APT.

Method	Q_{\min}^2	μ_4/M^2	μ_6/M^4	μ_8/M^6
The best μ_4 -fit results				
PT NLO	0.5	-0.028(5)	-	-
PT N^2 LO	0.66	-0.014(7)	-	-
PT N^3 LO	0.71	0.006(9)	-	-
The best $\mu_{4,6,8}$ -fit results				
PT NLO	0.27	-0.03(1)	-0.01(1)	0.008(4)
PT N^2 LO	0.34	0.01(2)	-0.06(4)	0.04(2)
PT N^3 LO	0.47	0.05(4)	-0.2(1)	0.12(6)

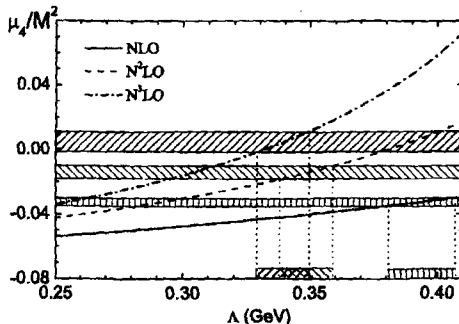


FIG. 5: Value of the higher twist coefficient μ_4 extracted from the JLab data using the PT at different orders at $Q_{min}^2 = 0.66 \text{ GeV}^2$ with error bands. Vertical lines denote the corresponding uncertainty ranges in Λ -parameter. The ranges corresponding to $N^2\text{LO}$ and $N^3\text{LO}$ approximations have similar sizes and overlap with each other, so the four-loop result does not improve the stability w.r.t. Λ variations compared to the three-loop one.

not lead to a stable result for extracted μ_4 value with respect to Λ variations. The extracted higher twist coefficient μ_4 changes quite strongly between different orders of the PT expansion. And it happens in both in absolute value and sign, namely, at $\Lambda > 320 \text{ MeV}$ the higher twist coefficient becomes positive in the four-loop PT order. This sensitivity of the higher twist term μ_4 to variations of the Λ becomes stronger at higher PT orders.

On the other hand, these data tell us that the absolute value of μ_4 decreases with the order of PT and just at four-loop order becomes compatible to zero. This may be considered as a manifestation of duality between higher orders of PT and HT (see Ref. [7] and references therein). Moreover, when PT series manifests the asymptotic behavior (i.e. becomes most close to exact result), the HT (which may be considered as a contribution completing the PT series) can be reduced to zero.

4. Summary and Conclusion

In this work, we presented the QCD analysis of the precise low energy JLab data on the BSR in the $N^3\text{LO}$ PT order and extracted the OPE higher twist terms using the four-loop expression for the QCD correction to the Bjorken integral Δ_{Bj} published recently in Ref. [8].

Our main observations are:

- i) The four-loop approximation provides good description of the data for the highest JLab $Q^2 \sim 3 \text{ GeV}^2$. For several data points there is an impression that the four-loop approximation is better than the three-loop one. At the same time, the order of magnitude of both these contributions is the same as an experimental error, so a more precise statement can hardly be made.
- ii) For lower $Q^2 \leq 0.7 \text{ GeV}^2$ the four-loop PT contribution does not help to describe the data.
- iii) The magnitude of HT decreases with an order of PT and becomes compatible to zero at the four-loop level.

Our concluding impression is that all these features may indicate that the asymptotic nature of the QCD PT series is revealed at the four-loop level at $Q^2 \sim 1 \text{ GeV}^2$. This asymptotic PT series leaves no room for higher twist correction when approaching the data most closely.

Acknowledgments

We are thankful to S.V. Mikhailov and K.G. Chetyrkin for valuable discussions as well as to R.S. Pasechnik for interest in the work and stimulating discussions.

This work was partly supported by the BelRFFR–JINR grant F10D-001 and the RFBR grant 11-01-00182, and the Russian presidential grant Scient. School–3810.2010.2.

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