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Threshold Resummation Factors in the Relativistic Quasipotential Approach in the Case of Unequal Masses

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The new relativistic of Coulomb-like threshold resummation factors in quantum chromodynamics in the case of $\ell \geq 0$ are received. Consideration is conducted within the framework of quasipotential approach in quantum theory of the field, formulated in the relativistic configurational representation for the case of two particles of unequal masses.

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1. Introduction

For the description of quark-antiquark systems close to threshold does not allow us to truncate the perturbative series even if the expansion parameter, the QCD coupling constant α_s, is small [1]. The reason consist in this, that the real expansion parameter in the threshold region is α/v, where $v = \sqrt{1-4m^2/s}$ is a quark velocity, and m is a quark mass. Obviously, it becomes to be singular, when the velocity $v \to 0$. The problem is well known from QED [2]. These threshold singularities of the form $(\alpha/v)^n$ have to be summarized. In the nonrelativistic of case, for the Coulomb interaction

$$V(r) = -\alpha/r \tag{1}$$

this resummation is realized by the known Gamov–Sommerfeld–Sakharov S-factor [3–5]

$$S_{\rm nr} = \frac{X_{\rm nr}}{1-\exp(-X_{\rm nr})}, \qquad X_{\rm nr} = \frac{\pi\,\alpha}{v_{\rm nr}}, \tag{2}$$

which is related to the wave function of the continuous spectrum at the origin by $|\psi(0)|^2$. Here $2\,v_{\rm nr}$ is the relative velocity of two nonrelativistic particles. The corresponding nonrelativistic expression can also be obtained for higher ℓ states (see, e.g., [6]).

In the relativistic theory the nonrelativistic approximation needs to be modified. The relativistic modification of the S-factor (2) in QCD in the case of two particles of equal masses ($m_1 = m_2 = m$) was executed in [7] (see also [8]) and it consisted in the change $v_{\rm nr} \to v$. Just the same form of the S-factor for the interaction of two particles of equal masses was later suggested in [9]. Another form of the relativistic generalization of the S-factor also in the case of two particles of equal masses was obtained in [10]. The relativistic S-factor for two particles of arbitrary masses ($m_1 \neq m_2$) was presented in [11].

The new approach to relativistic generalization of the S-factor in the case of two particles of equal masses was made by Milton and Solovtsov in [12]. For this aim has been used the quasipotential (QP) approach proposed by Logunov and Tavkhelidze [13], in the form suggested by

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Kadyshevsky [14]. In [12] has been used a transformation of the QP equation from momentum space into relativistic configurational representation (RCR) for the case of two particles equal masses [15]. Moreover, in [12] has been used the potential (1) considering in [16] that take into account its QCD-like behaviour. The solution containing arbitrary functions of r with period i, the so-called the i-periodic constants, with same potential have been investigated in [17]. However, the using of such solution is suitable only for the spectral problems.

Thus, in [12] has been made the new step to application of quasipotential approach in QCD. This approach gives the following expression for relativistic S-factor [12]:

$$S(\chi) = \frac{X(\chi)}{1-\exp\{-X(\chi)\}}, \qquad X(\chi) = \frac{\pi\,\alpha}{\sinh\chi} = \pi\alpha\sqrt{1-v^2}/v, \tag{3}$$

where χ is the rapidity related to the total c. m. energy of interacting particles, $\sqrt{s}$, by $2m\cosh\chi = \sqrt{s}$.

The S-factor appears in the parametrization of the imaginary part of the quark current correlator, the Drell ratio $R(s)$, which can be approximated in terms of the Bethe-Salpeter (BS) amplitude of two charged particles $\chi_{\rm BS}(x)$ at $x = 0$ [18]. The nonrelativistic replacement of this amplitude by the wave function, which obeys the Schrödinger equation with the Coulomb potential (1), leads to formula (2) with a substitution $\alpha \to 4\,\alpha_s/3$ for the QCD case. The possibility of using the QP approach for our task is based on the fact that the BS amplitude, which parameterizes the physical quantity $R(s)$, is taken at $x = 0$; therefore, in particular, at the relative time $\tau = 0$. Thus, the QP wave function in the momentum space is defined as the BS amplitude at $\tau = 0$ and, therefore, $R(s)$ can be expressed through the QP wave function in the momentum space, $\Psi_q(\mathbf{p})$, by using the relation

$$\chi_{\rm BS}(x=0) = \frac{1}{(2\pi)^3}\int d\Omega_p\,\Psi_q(\mathbf{p})\,, \tag{4}$$

where $d\Omega_p = (m\,d\mathbf{p})/E_p$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid $E_p^2 - \mathbf{p}^2 = m^2$.

The relativistic P-factor (for $\ell = 1$ state) in the case of two particles of equal masses was obtained in [19]. In that paper, a new model expression for $R(s)$, in which threshold singularities were summarized to the main potential contribution, was suggested as well. The generalization of the relativistic S- and P-factors for arbitrary ℓ states in the case of two particles of equal masses was executed in [20, 21]. This approach gives the following expression for relativistic L-factor [20, 21]:

$$L(\chi) = \frac{X(\chi)}{1-\exp[-X(\chi)]}\prod_{n=1}^{\ell}\left[1+\left(\frac{\alpha}{2n\sinh\chi}\right)^2\right]. \tag{5}$$

Applications of the factor (3) for describing some hadronic processes can be found in [22–24]. Recently, the relativistic S-factor (3) has been applied to reanalyze the mass limits obtained for magnetic monopoles which might have been produced at the Fermilab Tevatron [25].

The purpose of this paper is to generalize the previous study started in [12] to the case of the interaction of two particles of unequal masses ($m_1 \neq m_2$). We present the formalism of the relativistic QP approach in quantum field theory formulated in the relativistic configuration representation for the interaction of two relativistic particles of unequal masses. Within the framework of this approach we derive the relativistic factors in the case of $\ell \geq 0$. In the following we will use the system of units $c = \hbar = 1$.

2. The integral form of quasipotential equation for the case of two particles unequal masses

A starting point of our consideration is QP equation into the momentum space constructed in [26] for the QP wave function $\Psi_{q'}(\mathbf{p}')$ of two relativistic particles of unequal masses. In chosen system of units this equation is given by

$$(2E_{q'} - 2E_{p'})\,\Psi_{q'}(\mathbf{p}') = \frac{2\mu}{m'\,(2\pi)^3}\int d\Omega_{k'}\,\widetilde{V}\,(\mathbf{p}',\mathbf{k}';\,E_{q'})\,\Psi_{q'}(\mathbf{k}')\,, \tag{6}$$

where

$$d\Omega_{k'} = \frac{m'\,d\mathbf{k}'}{E_{k'}}$$

is the relativistic three-dimensional volume element in the Lobachevsky space, $E_{k'} = \sqrt{m'^2 + \mathbf{k}'^2}$, $m' = \sqrt{m_1 m_2}$, and $\mu = m_1 m_2/(m_1 + m_2)$ is the usual reduced mass.

The equation (6) represents a relativistic generalization of the Lippmann-Schwinger equation in the spirit of Lobachevsky geometry, which is realized on the upper half of the mass hyperboloid $E_{k'}^2 - \mathbf{k}'^2 = m'^2$, and describes the scattering of an effective relativistic particle of mass m' with a relative 3-momentum $\mathbf{k}'$ on the quasipotential $\widetilde{V}\,(\mathbf{p}',\,\mathbf{k}';\,E_{q'})$, the total c. m. energy of the particles involved being proportional to the energy of one effective relativistic particle of mass m' [26, 27]:

$$\sqrt{s} = \sqrt{m_1{}^2 + \mathbf{k}^2} + \sqrt{m_2{}^2 + \mathbf{k}^2} = \frac{m'}{\mu}\sqrt{m'^2 + \mathbf{k}'^2}\,. \tag{7}$$

We note that Eq. (6) differs from of the QP equation considered in [28] by means of introduction into it of the relativistic reduced mass. However, in [28] was shown that it is possible to use the different expressions for the relativistic reduced mass by means of the choice of functional relationship between the relative 3-momentum $\mathbf{k}$ of the interacting particles and their the relativistic relative velocity, v, which is expressed through their the total energy of c. m. particles, $\sqrt{s}$, by relation (see, for instance, [10, 11])

$$\mathrm{v} = 2\sqrt{\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}}\left(1+\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}\right)^{-1}\,. \tag{8}$$

In particular, if the dependence between the relativistic relative velocity v and the square of relative 3-momentum $\mathbf{k}'$ for an effective relativistic particle, having mass m' and emerging instead of the system of two particles, is given by expression (see [26, 27])

$$\mathbf{k}'^2 = 2\mu^2\left(\frac{1}{\sqrt{1-\mathrm{v}^2}} - 1\right)\,, \tag{9}$$

this lead to the equation (6).

The proper Lorentz transformations means a translation in the Lobachevsky space. The role of the plane waves corresponding to these translations are played by the following functions:

$$\xi(\mathbf{p}',\mathbf{r}) = \left(\frac{E_{p'} - \mathbf{p}'\cdot\mathbf{n}}{m'}\right)^{-1-irm'}\,, \tag{10}$$

where the module of the radius-vector, $\mathbf{r}$, ($\mathbf{r} = r\,\mathbf{n}\,,|\mathbf{n}| = 1$) is a relativistic invariant [27]. These functions correspond to the principal series of unitary representations of the Lorentz group and in the nonrelativistic limit ($p' \ll 1$, $r \gg 1$) $\xi(\mathbf{p}', \mathbf{r}) \to \exp(i\mathbf{p}' \cdot \mathbf{r})$. The functions (10) obey the following conditions of completeness and orthogonality [27]

$$\frac{1}{(2\pi)^3} \int d\Omega_{p'}\, \xi(\mathbf{p}', \mathbf{r})\, \xi^*(\mathbf{p}', \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad \frac{1}{(2\pi)^3} \int d\mathbf{r}\, \xi(\mathbf{p}', \mathbf{r})\, \xi^*(\mathbf{q}', \mathbf{r}) = \delta(\mathbf{p}'(-)\mathbf{q}'), \tag{11}$$

and these the functions satisfy the equation in terms of finite differences

$$\left(2E_{p'} - \hat{H}_0\right) \xi(\mathbf{p}', \mathbf{r}) = 0. \tag{12}$$

Here $\delta(\mathbf{p}'(-)\mathbf{q}') = \sqrt{1 + \mathbf{p}'^2/m'^2}\, \delta(\mathbf{p}' - \mathbf{q}')$ is the relativistic δ-function in momentum-space, the operator

$$\hat{H}_0 = 2\, m' \left[\cosh\left(i\lambda' \frac{\partial}{\partial r}\right) + \frac{i\lambda'}{r} \sinh\left(i\lambda' \frac{\partial}{\partial r}\right) - \frac{\lambda'^2\, \Delta_{\theta,\varphi}}{2r^2} \exp\left(i\lambda' \frac{\partial}{\partial r}\right)\right] \tag{13}$$

is the operator of the free Hamiltonian, while $\Delta_{\theta,\varphi}$ is its the angular part and $\lambda' = 1/m'$ is the Compton wavelengs associated with the effective particle of mass m'.

The QP wave functions in the momentum space and RCR are related as follows [26, 27]:

$$\psi_{q'}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\Omega_{p'}\, \xi(\mathbf{p}', \mathbf{r})\, \Psi_{q'}(\mathbf{p}'), \qquad \Psi_{q'}(\mathbf{p}') = \int d\mathbf{r}\, \xi^*(\mathbf{p}', \mathbf{r})\, \psi_{q'}(\mathbf{r}). \tag{14}$$

For a spherically symmetric potential the application of transformations (14) (Shapiro transformations or ξ-transformations) to Eq. (6) leads to the equation which is the integral form of the relativistic Schrödinger equation in the configuration representation:

$$\frac{1}{(2\pi)^3} \int d\Omega_{p'}\, (2E_{q'} - 2E_{p'})\, \xi(\mathbf{p}', \mathbf{r}) \int d\mathbf{r}'\, \xi^*(\mathbf{p}', \mathbf{r}')\, \psi_{q'}(\mathbf{r}') = \frac{2\mu}{m'} V(r)\, \psi_{q'}(\mathbf{r}), \tag{15}$$

where the right hand side already is local in RCR, and the transform of the potential, $V(r)$, is given in terms of the same relativistic plane waves.

We note that the using of relations (14) and the equation (12) allows us to express the left hand side of equation (15) in terms finite differences

$$\left(2E_{q'} - \hat{H}_0\right) \psi_{q'}(\mathbf{r}) = \frac{2\,\mu}{m'} V(r)\, \psi_{q'}(\mathbf{r}). \tag{16}$$

Solutions of this equation, in principle, can contain arbitrary functions of r with period i, the so-called the i-periodic constants, which appear in the solutions due to the finite difference nature of the Hamiltonian (13). For some problems, such as defining the bound state spectrum, this i-periodic constant is not important. However, for the purpose of extracting resummation factors, one must develop a method which avoids this ambiguity. For this instead of the equation (16) we will to use the equation (15).

The integral equation (15) can be reduced to the form

$$\frac{1}{(2\pi)^3} \int d\Omega_{p}\, (2E_{q} - 2E_{p})\, \xi(\mathbf{p}, \boldsymbol{\rho}) \int d\boldsymbol{\rho}'\, \xi^*(\mathbf{p}, \boldsymbol{\rho}')\, \psi_{q}(\boldsymbol{\rho}') = \frac{2\,\mu}{m'} V(\rho)\, \psi_{q}(\boldsymbol{\rho}), \tag{17}$$

where we introduced the following notation:

$$\begin{gathered}
\mathbf{q}' = m'\,\mathbf{q},\ \mathbf{p}' = m'\,\mathbf{p},\ \mathbf{q} = \sinh(\chi_q)\,\mathbf{n}_q,\ \mathbf{p} = \sinh(\chi_p)\,\mathbf{n}_p,\ |\mathbf{n}_q| = |\mathbf{n}_p| = 1, \\
\boldsymbol{\rho} = m'\,\mathbf{r},\ \boldsymbol{\rho}' = m'\,\mathbf{r}',\ \rho = m'\,r,\ \rho' = m'\,r', \\
d\mathbf{r}' = m'^{-3}\, d\boldsymbol{\rho}',\ d\Omega_{p'} = m'^3\, d\Omega_p,\ d\Omega_p = \frac{d\mathbf{p}}{E_p},\ E_{q'} = m'\,E_q,\ E_{p'} = m'\,E_p, \\
E_q = \sqrt{1 + \mathbf{q}^2},\ E_p = \sqrt{1 + \mathbf{p}^2},\ \xi(\mathbf{p}', \mathbf{r}) = \left(E_p - \mathbf{p} \cdot \mathbf{n}\right)^{-1-i\rho} \equiv \xi(\mathbf{p}, \boldsymbol{\rho}), \\
V(r) = V(\rho/m') \equiv m'\,V(\rho),\ \psi_{q'}(\mathbf{r}) = \psi_{m'q}(\boldsymbol{\rho}/m') \equiv \psi_q(\boldsymbol{\rho}),\ \Psi_{q'}(\mathbf{p}') \equiv m'^{-3}\, \Psi_q(\mathbf{p}).
\end{gathered} \tag{18}$$

By using of the expandings

$$\xi(\mathbf{p},\rho)=\sum_{\ell=0}^{\infty}(2\ell+1)\,i^{\ell}\,p_{\ell}(\rho,\cosh\chi_{p})\,P_{\ell}\Big(\frac{\mathbf{p}\cdot\boldsymbol{\rho}}{p\,\rho}\Big),$$

$$\psi_{q}(\rho)=\sum_{\ell=0}^{\infty}(2\ell+1)\,i^{\ell}\,\frac{\varphi_{\ell}(\rho,\chi)}{\rho}\,P_{\ell}\Big(\frac{\mathbf{q}\cdot\boldsymbol{\rho}}{q\,\rho}\Big), \tag{19}$$

and also formula [15]

$$p_{\ell}(\rho,\cosh\chi)=\frac{(-1)^{\ell}(\sinh\chi)^{\ell}}{\rho^{(\ell+1)}}\Big(\frac{d}{d\cosh\chi}\Big)^{\ell}\Big(\frac{\sin\rho\chi}{\sinh\chi}\Big),$$

the equation (17) transformed to the form

$$\frac{2}{\pi}\int_{0}^{\infty}d\chi'\frac{(\sinh\chi')^{2\ell+2}(-1)^{\ell+1}}{\rho^{(\ell+1)}}(2\cosh\chi-2\cosh\chi')\Big(\frac{d}{d\cosh\chi'}\Big)^{\ell}\Big(\frac{\sin\rho\chi'}{\sinh\chi'}\Big)$$

$$\times\Big(\frac{d}{d\cosh\chi'}\Big)^{\ell}\frac{1}{\sinh\chi'}\int_{0}^{\infty}d\rho'\frac{\rho'\sin\rho'\chi'}{(-\rho')^{(\ell+1)}}\varphi_{\ell}(\rho',\chi)=\frac{2\,\mu}{m'}\frac{V(\rho)\,\varphi_{\ell}(\rho,\chi)}{\rho}. \tag{20}$$

Here

$$p_{\ell}(\rho,\cosh\chi)=\frac{(-1)^{\ell+1}}{\rho}\sqrt{\frac{\pi}{2\sinh\chi}}\,(-\rho)^{(\ell+1)}\,P^{-1/2-\ell}_{-1/2+i\rho}(\cosh\chi) \tag{21}$$

is the solution of equation (16) in the case when the interaction is switched off ($V(r)\equiv 0$); $P^{\nu}_{\mu}(z)$ is a Legendre function of the first kind; χ',χ are the rapidities which related to E_p,E_q by relations $E_p=\cosh\chi',E_q=\cosh\chi$, and the function

$$(-\rho)^{(\ell+1)}=i^{\ell+1}\frac{\Gamma(i\rho+\ell+1)}{\Gamma(i\rho)} \tag{22}$$

is the generalized power [15], where $\Gamma(z)$ is gamma-function.

Thus, Eq. (20) differs from the corresponding equation in the case of two particles of equal masses (see [21]) only by the factor $2\mu/m'$ turning into 1 at $m_1=m_2$.

3. Relativistic threshold resummation factors

To solve quasipotential equation (20) with the potential (1), we use the method developed in [12, 19, 29]. A solution of quasipotential equation (20) with the potential (1) one can seek in the form

$$\varphi_{\ell}(\rho,\chi)=\frac{(-\rho)^{(\ell+1)}}{\rho}\int_{\alpha_-}^{\alpha_+}d\zeta\,e^{i\rho\zeta}\,R_{\ell}(\zeta,\chi), \tag{23}$$

where the ζ-integration is performed in the complex plane over a contour with end points α_- and α_+ as in [12, 19, 29]: $\alpha_-=-R-i\varepsilon,\alpha_+=-R+i\varepsilon$ with $R\to+\infty,\varepsilon\to+0$. Substituting (23) into (20) and taking into account that

$$\frac{1}{i\pi}\int_{0}^{\infty}d\rho'\sin(\rho'\chi')\,e^{i\rho'\zeta}=\frac{1}{i\pi}\frac{\chi'}{\chi'^2-\zeta^2},$$

we arrive at the equation

$$(-1)^{\ell}\int_{\alpha_-}^{\alpha_+} d\zeta\, R_\ell(\zeta,\chi)\left(\frac{d}{d\cosh\zeta}\right)^{\ell}\Bigg[(\sinh\zeta)^{2\ell+1}\,(2\cosh\chi-2\cosh\zeta)$$
$$\times\left(\frac{d}{d\cosh\zeta}\right)^{\ell}\left(\frac{e^{i\rho\zeta}}{\sinh\zeta}\right)\Bigg]=-\frac{2\,\alpha\,\mu}{m'\,\rho}\prod_{n=1}^{\ell}(\rho^2+n^2)\int_{\alpha_-}^{\alpha_+} d\zeta\, e^{i\rho\zeta}\,R_\ell(\zeta,\chi)\,. \tag{24}$$

Follows to note that solutions of this equation, but signifies and equation (20), already does not contain the i-periodic constants which appear in the solutions (16) due to the finite difference nature of the Hamiltonian (13). For the first time, the solution of the equation (20) in the case of two particles equal masses at $\ell=0$, not containing the i-periodic function, was obtained *in [12]. This approach leads to the relativistic S-factor (3).*

The solution of Eq. (24) for $\ell\geq 0$ leads to the following expression for QP partial wave function:

$$\varphi_\ell(\rho,\chi)=-C_\ell(\chi)\frac{\rho}{(-1)^{\ell+1}\rho^{(\ell+1)}}\int_{\alpha_-}^{\alpha_+} d\zeta\frac{e^{(\ell+1+i\rho)\zeta}}{(e^{\zeta}-e^{\chi})^{2\ell+2}}\left[\frac{e^{\zeta}-e^{-\chi}}{e^{\zeta}-e^{\chi}}\right]^{-\ell-1+iA}, \tag{25}$$
$$A=\frac{\alpha\,\mu}{m'\,\sinh\chi}\,.$$

Performing in (25) ζ-integration in the complex plane along a contour with end points α_- and α_+ (in the same way as in [12, 19, 29]) one obtain the resulting solution which does not contain the i-periodic constant in the form

$$\varphi_\ell(\rho,\chi)=-C_\ell(\chi)\frac{2\rho\,\sinh(\pi\,\rho)}{\rho^{(\ell+1)}}\int_{-\infty}^{\infty} dx\frac{e^{(\ell+1+i\rho)x}}{(e^{x}+e^{\chi})^{2\ell+2}}\left[\frac{e^{x}+e^{-\chi}}{e^{x}+e^{\chi}}\right]^{-\ell-1+iA}. \tag{26}$$

The function (26) can also be represented in terms of hypergeometrical function as

$$\varphi_\ell(\rho,\chi)=N_\ell(\chi)\,(-\rho)^{(\ell+1)}\,e^{i\rho\chi+iA\chi+i\pi(\ell+1)}\,F\left(\ell+1-iA\,,\ell+1-i\rho\,;2\ell+2\,;1-e^{-2\chi}\right). \tag{27}$$

The normalization constant $N_\ell(\chi)=-\dfrac{2\pi\,C_\ell(\chi)}{\Gamma(2\ell+2)}\exp[-(\ell+1)\chi-iA\chi]$ in Eq. (27) can be obtained (also as in [21]) from the condition

$$\lim_{\alpha\to 0}\varphi_\ell(\rho,\chi)=\rho\,p_\ell(\rho,\cosh\chi)\xrightarrow[\rho\to\infty]{}\frac{\sin(\rho\chi-\pi\,\ell/2)}{\sinh\chi}, \tag{28}$$

where the function $p_\ell(\rho,\cosh\chi)$ it is determined in (21).

We should like to remind that the Bethe-Salpeter amplitude $\chi_{\rm BS}(x=0)$ is associated with the QP wave function in the momentum space, $\Psi_q(\mathbf{p})$, by the relation (4). Taking into consideration the transformations (14) and the notation (18), the relationship of the Bethe-Salpeter amplitude with QP wave function in RCR, $\psi_q(\rho)$, is of the form of

$$\chi_{\rm BS}(x=0)=\psi_q(\rho)|_{\rho=i}\,.$$

The generalized power (22) in the solution (27) vanishes at $\rho=i$ for all $\ell\neq 0$. Thus, the expansion (19) for the wave function $\psi_q(\rho)$ contains only s-wave ($\ell=0$). The L-factor in the nonrelativistic case is defined by the derivative of order $\ell > 0$ of the partial wave function corresponding $\ell > 0$ state at $r = 0$. In the relativistic case, instead of the derivative, we has to use its finite difference analog [27]

$$\Delta^* = \frac{1}{i}\left[\exp\left(i\frac{\partial}{\partial\rho}\right) - 1\right] . \tag{29}$$

Thus, the relativistic L-factor is connected, as one can expect, with QP partial wave function $\varphi_\ell(\rho\,,\chi)$ ($\ell \geq 0$) and it is defined by

$$L_{\rm uneq}(\chi) = \lim_{\rho\to i}\left|\frac{\Gamma(2\ell+2)}{(2\sinh\chi)^\ell\,\Gamma^2(\ell+1)}\,(\Delta^*)^\ell\left[\frac{\varphi_\ell(\rho\,,\chi)}{\rho}\right]\right|^2 . \tag{30}$$

Hence, by using relations (27)–(30) we can calculate $|(\Delta^*)^\ell\,[\varphi_\ell(\rho\,,\chi)/\rho]\,|^2$ at $\rho = i$, which leads to the following expression for the relativistic L-factor in the case of two particles of unequal masses:

$$L_{\rm uneq}(\chi) = \frac{X_{\rm uneq}(\chi)}{1-\exp[-X_{\rm uneq}(\chi)]}\prod_{n=1}^{\ell}\left[1+\left(\frac{A}{n}\right)^2\right], \qquad X_{\rm uneq}(\chi) = \frac{2\,\pi\,\alpha\,\mu}{m'\,\sinh\chi}, \tag{31}$$

where χ is the rapidity which related to the center of mass energy, $\sqrt{s}$, by $(m'^2/\mu)\,\cosh\chi = \sqrt{s}$.

The function $X_{\rm uneq}(\chi)$ and parameter A in Eq. (31) can be expressed in terms of the "velocity" u determined by relation

$$u = \sqrt{1-\frac{4m_1\,m_2}{\bar{s}}}, \qquad \bar{s} = s - (m_1 - m_2)^2\,, \tag{32}$$

in the form

$$X_{\rm uneq}(\chi) = \frac{\pi\,\alpha\,\sqrt{1-u^2}}{u}, \quad A = \frac{\alpha\,\sqrt{1-u^2}}{2u}. \tag{33}$$

We note that the square of relative 3-momentum $\mathbf{k}'$ for an effective relativistic particle, having mass m', the total c. m. energy of interacting particles, $\sqrt{s}$, and emerging instead of the system of two particles, is defined by formula (7) and connected with the relative relativistic velocity of interacting particles, v, by the expression (9). In turn the relative velocity v of interacting particles is expressed through their the total of c. m. energy of interacting particles, $\sqrt{s}$, by relation (8). Thence, taking into consideration the determination (32), we find

$$\mathrm{v} = \frac{2\,u}{1+u^2}. \tag{34}$$

Then expressions (9) and (34) give

$$\mathbf{k}'^2 = (\mu'_{\rm rel})^2\,(u'_{\rm rel})^2\,, \tag{35}$$

where $\mu'_{\rm rel} = 2\,\mu$ is the relativistic reduced mass, and

$$u'_{\rm rel} = \frac{u}{\sqrt{1-u^2}} \tag{36}$$

is the relative velocity of an effective relativistic particle of mass m'. This result is found to be in full agrement with the physical sense of Eq (6), being a relativistic generalization of the Lippmann-Schwinger equation in the spirit of Lobachevsky geometry. This equation describes the scattering of an effective relativistic particle of mass m' on the quasipotential $\widetilde{V}\,(\mathbf{p}',\,\mathbf{k}';\;E_{q'})$.

The effective relativistic particle emerges instead of the system of two particles, has mass m', the relative 3-momentum $\mathbf{k}'$ and carries the total c. m. energy of the interacting particles, $\sqrt{s}$. Notice that the 3-momentum $\mathbf{k}'$ of an effective relativistic particle and hence its relative velocity (36), according to Eqs. (9), and (35), are an invariants of the Lorentz transformations.

Thus, in the terms of relative velocity of an effective relativistic particle the L-factor (31) is given by the expression

$$L_{\rm uneq}(u^{'}_{\rm rel}) = \frac{X_{\rm uneq}(u^{'}_{\rm rel})}{1-\exp\left[-X_{\rm uneq}(u^{'}_{\rm rel})\right]}\prod_{n=1}^{\ell}\left[1+\left(\frac{\alpha}{2\,n\,u^{'}_{\rm rel}}\right)^2\right]\,,\; X_{\rm uneq}(u^{'}_{\rm rel}) = \frac{\pi\,\alpha}{u^{'}_{\rm rel}}\,. \qquad (37)$$

The relativistic threshold resummation factor (31) [or (37)] at $\ell=0$ coincides with S-factor obtained in [30] and have the following important properties.

- In the nonrelativistic limit, $u \ll 1$, it reproduces the known nonrelativistic result.
- In the case of equal masses it coincides with L-factor (5) [21].
- In the ultrarelativistic limit, as it has been argued in [31, 32], the bound state spectrum vanishes as mass of an effective relativistic particle $m' \to 0$. This feature reflects an essential difference between potential models and quantum field theory, where an additional dimensional parameter appears. One can conclude that within a potential model, the L-factor which correspond to the continuous spectrum should go to unity in the limit $m' \to 0$. Thus, in contrast to the nonrelativistic case, the relativistic resummation factor, the L-factor (31) [or (37)], reproduce both the known nonrelativistic and the expected ultrarelativistic limits.
- In the relativistic limit, $u \to 1$, the L-factor go to unity.

We note that this new relativistic factor could have a significant impact in interpreting strong-interaction physics. In many physically interesting cases, the function $R(s)$ occurs as a factor in an integrand, as, for example, for the case of inclusive τ decay, for smearing quantities, and for the Adler D function. Here the behaviour of L-factor at intermediate values of u becomes important.

4. Conclusion

In this paper, the relativistic of Coulomb-like threshold resummation factor (37) in the case of $\ell \geq 0$ for the interaction of two relativistic particles of unequal masses was obtained. For this aim the relativistic quasipotential equation in relativistic configuration representation [26] with the Coulomb potential for the interaction of two relativistic particles of unequal masses was used. The Coulomb potential only formally has the same form as the nonrelativistic potential but differs in the relativistic configuration representation since its behavior corresponds to the quark-antiquark potential $V_{q\bar{q}} \sim \bar{\alpha}_s(Q^2)/Q^2$ with the invariant charge $\bar{\alpha}_s(Q^2) \sim 1/\ln Q^2$. So, the principal effect coming from the running of the QCD coupling is accumulated.

The new relativistic L-factor (37) reproduces both the known nonrelativistic behavior and the expected ultrarelativistic limit. The new L-factor at $\ell=0$ coincides in form with the nonrelativistic S-factor (2); however, the role of the parameter of velocity is played not by the relative velocity of interacting particles, v, but by the relative velocity (36) of an effective relativistic particle emerging instead of the system of two particles. As the new relativistic resummation factor (37) was obtained within the framework of completely covariant method, one can expect that this factor takes into account more adequately relativistic nature of interaction.

The new relativistic resummation factor can have a significant impact in interpreting strong-interaction physics since it has the influence on the behavior of the Drell ration $R(s)$. In some physically interesting cases the function $R(s)$ occurs as a factor in an integrand, as, for example, in the case of inclusive τ decay or in the Adler D-function, and the behavior of the L-factor at intermediate values of variable u becomes important.

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