# Nonlocal Separable Interaction and Relativistic Finite-Difference Quasipotential Equation

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About forty years ago Logunov and Tavkhelidze proposed the quasipotential approach to the relativistic two-body problem [1]. The aim of this paper is to give a summary of the method developed to solve the finite-difference quasipotential equation. We are considering a nonlocal separable interaction quasipotential for two relativistic spinless particles with nonequal masses ($m_1 \neq m_2$) in the framework of relativistic quasipotential approach [2]. On the basis of solutions found by means of the method, the nonlocal separable interaction quasipotential of two relativistic spinless particles with nonequal masses is reconstructed using the phase shifts and the mass spectrum. Our consideration is based on the finite-difference equation for the relativistic wave function $\Psi_{q'}(\vec{r})$ in $\vec{r}$ -representation [3] ($\hbar = c = 1$):

$$\left(\sqrt{S_{q'}} - H_0\right)\Psi_{q'}(\vec{r}) = \int d\vec{r'} V(\vec{r}, \vec{r'}; E_{q'})\Psi_{q'}(\vec{r}), \tag{1}$$

$$H_0 = \frac{m'^2}{\mu}\left[\mathrm{ch}\left(\frac{i\lambda'\partial}{\partial r}\right) + \frac{i\lambda'}{r}\mathrm{sh}\left(\frac{i\lambda'\partial}{\partial r}\right) - \frac{\lambda'^2}{2r^2}\Delta_{\theta,\varphi}\exp\left(\frac{i\lambda'\partial}{\partial r}\right)\right], \tag{2}$$

$$V(\vec{r}, \vec{r'}; E_{q'}) = (2\pi)^{-6}\int d\Omega_{p'} d\Omega_{k'} \xi(\vec{p'}, \vec{r})\tilde{V}(\vec{p'}, \vec{k'}; E_{q'})\xi^*(\vec{k'}, \vec{r'}), \tag{3}$$

where $d\Omega_{k'} = d\vec{k'}/\sqrt{1 + (\vec{k'}/m')^2}$ is the invariant element of volume in the Lobachevsky space taken on the upper sheet of the hyperboloid $k'^2 = m'^2$, $\sqrt{S_{k'}} = \sqrt{m_1^2 + \vec{k}^2} + \sqrt{m_2^2 + \vec{k}^2} = \frac{m'}{\mu}\sqrt{m'^2 + \vec{k}'^2}$ is the total energy of particles in the c.m.system, $\vec{k}'$ and $m' = \sqrt{m_1 m_2}$ are the momentum and mass of the effective relativistic particle,

$\xi(\vec{p'}, \vec{r}) = \left(\frac{p_0' - \vec{p}' \cdot \vec{n}}{m'}\right)^{-1-irm'}$ are the relativistic plane waves, $p_0' = E_{p'} = \sqrt{m'^2 + \vec{p'}^2}$, $\vec{r} = r\vec{n}$, $|\vec{n}| = 1$ is the relativistic radius vector, $\mu = \frac{m'^2}{m_1 + m_2}$, $\Delta_{\theta,\varphi}$ is the angular part of the Laplas operator, $\lambda' = 1/m'$, and $\tilde{V}(\vec{p'}, \vec{k'}; E_{q'})$ is the quasipotential.

The representation for the separable quasipotential is defined in the form

$$V(\vec{r}, \vec{r'}; E_{q'}) = V(\vec{r}, \vec{r'}) = \sum_{l=0}^{\infty}(2l+1)\varepsilon_l \upsilon_l(r)\upsilon_l(r')P_l\left(\frac{\vec{r} \cdot \vec{r'}}{rr'}\right), \tag{4}$$

and the wave function $\Psi_{q'}(\vec{r})$ has the decomposition

$$\Psi_{q'}(\vec{r}) = \sum_{l=0}^{\infty}(2l+1)i^l \frac{\varphi_l(\chi', r)}{r}P_l\left(\frac{\vec{q'} \cdot \vec{r}}{q'r}\right), \tag{5}$$

where $P_l(z)$ are the Legendre functions of the first type. Then instead of the equation (1) we obtain for the radial wave function $\varphi_l(\chi', r)$ the equation

$$\left[\mathrm{ch}\left(i\lambda'\frac{d}{dr}\right) + \frac{\lambda'^2 l(l+1)}{2r^{(2)}}\exp\left(\frac{i\lambda' d}{dr}\right) - \mathrm{ch}\chi'\right]\varphi_l(\chi', r) + \frac{1}{2}\varepsilon_l V_l(r)\int_0^{\infty} dr' V_l(r')\varphi_l(\chi', r') = 0 \tag{6}$$

with the boundary condition

$$\varphi_l(\chi', 0) = 0. \tag{7}$$

Here $\sqrt{S_{q'}} = \frac{m'^2}{\mu}\sqrt{1+(\vec{q}'/m')^2} = \frac{m'^2}{\mu}\mathrm{ch}\chi'$, $V_l(r) = \sqrt{8\pi\mu/m'^2}\,rv_l(r)$, $\varepsilon_l = \pm 1$, $r^{(2)} = r(r+i\lambda')$, and the quasipotential $V_l(r)$ satisfies the condition

$$rV_l(r) \in L_1(0,\infty). \tag{8}$$

Let us introduce the relativistic integral Hankel transoformations:

$$\tilde{\varphi}_l(\chi',\chi) = \int\limits_0^\infty dr\varphi_l(\chi',r)S_l^*(\chi,r)/Q_l(\mathrm{cth}\chi), \tag{9}$$

$$\varphi_l(\chi,r) = \frac{2}{\pi}\int\limits_0^\infty d\chi Q_l(\mathrm{cth}\chi)\tilde{\varphi}_l(\chi',\chi)S_l(\chi,r),$$

$$\tilde{V}_l(\chi) = \int\limits_0^\infty drV_l(r)S_l^*(\chi,r)/Q_l(\mathrm{cth}\chi),$$

$$V_l(r) = \frac{2}{\pi}\int\limits_0^\infty d\chi Q_l(\mathrm{cth}\chi)\tilde{V}_l(\chi)S_l(\chi,r).$$

Here $Q_l(z)$ are the Legendre functions of the second type. Using the transformation (9), the solution of equation (6) for the scattering states can be obtained in the form

$$\varphi_l(\chi',r) = \frac{S_l(\chi',r)}{Q_l(\mathrm{cth}\chi')} + \frac{1}{\pi}\varepsilon_l N_l(\chi')\mathrm{P}\int\limits_0^\infty d\chi Q_l(\mathrm{cth}\chi)\frac{\tilde{V}_l(\chi)S_l(\chi,r)}{\mathrm{ch}\chi'-\mathrm{ch}\chi}, \tag{10}$$

$$N_l(\chi') = \tilde{V}_l^*(\chi')\left[1+\mathrm{P}\frac{1}{2}\int\limits_0^\infty d\chi\frac{A_l(\chi)}{\mathrm{ch}\chi-\mathrm{ch}\chi'}\right]^{-1}, \tag{11}$$

$$A_l(\chi) = \frac{2}{\pi}\varepsilon_l Q_l^2(\mathrm{cth}\chi)\left|\tilde{V}_l(\chi)\right|^2, \tag{12}$$

where P is the value principal.

Here $S_l(\chi',r)$ is the solution of equation (6) at $\varepsilon_l = 0$. Using the asymptotic behavior of function (10) we obtain the following expression for the phase shift $\delta_l(\chi')$:

$$\mathrm{tg}\delta_l(\chi') = -\frac{\pi}{2}\mathrm{sh}^{-1}\chi' A_l(\chi')\left[1+\mathrm{P}\frac{1}{2}\int\limits_0^\infty d\chi\frac{A_l(\chi)}{\mathrm{ch}\chi-\mathrm{ch}\chi'}\right]^{-1}, \tag{13}$$

where $\delta_l(\infty) = 0$.

The energies of bound states $0 \le E' = E'_i/m' = \mathrm{ch}\chi'_i = \cos\kappa'_i < 1 \quad (\chi'_i = i\kappa'_i)$ are determined by the condition

$$1+\frac{1}{2}\mathrm{P}\int\limits_0^\infty\frac{d\chi A_l(\chi)}{\mathrm{ch}\chi - E'} = 0, \tag{14}$$

and the energies of scattering states $E' = E'_R/m' = \mathrm{ch}\chi'_R \ge 1$ are determined by the conditions (14) and (15):

$$\tilde{V}_l(\chi'_R) = 0, \quad \varepsilon_l = \pm 1. \tag{15}$$

If we now solve the integral equation (13) we get

$$\begin{aligned} A_l(\chi') &= -(2/\pi)\mathrm{sh}\chi'\sin\delta_l(\chi')\exp\left[\alpha_l(\mathrm{ch}\chi')\right]\times \\ &\times\prod_{n=0}^{\nu_l-1}\left(1-\frac{\mathrm{ch}\chi'_{Rn}-1}{\mathrm{ch}\chi'-1}\right)\prod_{k=0}^{n_l-1}\left(1+\frac{1-\cos\kappa'_{ik}}{\mathrm{ch}\chi'-1}\right), \end{aligned} \tag{16}$$

$$\alpha_l(\mathrm{ch}\chi') = -(1/\pi)\mathrm{P}\int_0^\infty d\chi \mathrm{sh}\chi\delta_l(\chi)/(\mathrm{ch}\chi - \mathrm{ch}\chi'), \tag{17}$$

where $n_l$ is the number of bound states with energies $0 \leq E'_{ik} = m' \cos\kappa'_{ik} < m'$, $\chi'_{ik} = i\kappa'_{ik}$, $k = 0, 1, \ldots, n_l - 1$, and $\nu_l$ is the number of scattering states with energies $E'_{Rn} = m'\mathrm{ch}\chi'_{Rn} \geq m'$, $n = 0, 1, 2, \ldots, \nu_l - 1$. Using the Hilbert integral transformations of the real and imaginary parts of the analytical function $\ln \hat{V}_l(\chi'), \mathrm{Im}\chi' > 0$, where

$$\hat{V}_l(\chi') = \prod_{k=0}^{n_l-1} \left(\frac{\chi' + i\kappa'_{ik}}{\chi' - i\kappa'_{ik}}\right) Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')/A_l^{as}, \tag{18}$$

we reconstruct the quasipotential in the form

$$Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi') = A_l^{as} \prod_{k=0}^{n_l-1} \left(\frac{\chi' - i\kappa'_{ik}}{\chi + i\kappa'_{ik}}\right) \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\chi \frac{\ln(\pi\varepsilon_l A_l(\chi)/2(A_l^{as})^2)}{\chi - \chi'}\right\} \tag{19}$$

in the region $\mathrm{Im}\chi' > 0$.
Here $A_l^{as}$ is the asymptotic behaviour of the function

$$\left|Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')\right| = \sqrt{(\pi/2)\varepsilon_l A_l(\chi')}, \qquad \mathrm{when}|\chi'| \to \infty. \tag{20}$$

As a result we obtained the solutions of direct and inverse problems in the framework of relativistic quasipotential approach. We considered the case of nonlocal separable interaction quasipotential between two relativistic particles without spin and with nonequal masses.

## References


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[3] V.G. Kadyshevsky, M.D. Mateev, R.M. Mir-Kasimov. Nucl. Phys. 1970. V11. p.692.