Study of the Perturbative QCD Series Structure to Order α_s^4

V. Khandramai[*], *O. Solovtsova*
Gomel State Technical University, 246746 Gomel, Belarus
D. Shirkov, O. Teryaev
Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russia
R. Pasechnik
High Energy Physics, Department of Physics and Astronomy, Uppsala University, SE-75121 Uppsala, Sweden

Abstract

We study the polarized Bjorken sum rule at low momentum transfers in the range $0.22 < Q < 1.73$ GeV with the four-loop N^{3}LO expression for the coefficient function $C_{\mathrm{Bj}}(\alpha_{\mathrm{s}})$ in the framework of the common QCD perturbation theory (PT) and the singularity-free analytic perturbation theory (APT). The analysis of the PT series for $C_{\mathrm{Bj}}(\alpha_{\mathrm{s}})$ gives a hint to its asymptotic nature manifesting itself in the region $Q < 1$ GeV. It relates to the observation that the accuracy of both the three- and four-loop PT predictions happens to be at the same 10% level. On the other hand, the usage of the two-loop APT allows one to describe the precise low energy JLab data down to $Q \sim 300$ MeV and gives a possibility for reliable extraction of the higher twist (HT) corrections. At the same time, above $Q \sim 700$ MeV the APT two-loop order with HT is equivalent to the four-loop PT with HT compatible to zero and is adequate to current accuracy of the data.

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[*] E-mail:v.khandramai@gmail.com

1 Introduction

The most important and fundamental sum rule for polarized deep inelastic scattering (DIS) is the Bjorken sum rule (BSR) which was derived in 1966 from current algebra [1]. This sum rule is so very fundamental because it relies only on isospin invariance, i.e. on a SU(2) symmetry between u and d quarks. The Bjorken integral has been measured in polarized deep inelastic lepton scattering at SLAC [2, 3, 4, 5], CERN [6, 7], DESY [8], and recently at low Q^2, $0.05 < Q^2 < 3\,\mathrm{GeV}^2$, in the CEBAF at Jefferson Lab (JLab) [9].

The BSR claims that the difference of the proton and neutron structure functions integrated over all values

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 \left[g_1^p(x,Q^2) - g_1^n(x,Q^2)\right] dx\,, \tag{1}$$

of the Bjorken variable x in the limit of large four-momentum squared of the exchanged virtual photon, $Q^2 \to \infty$, is equal to $g_A/6$, with $g_A = 1.267 \pm 0.004$ [10], the nucleon axial charge defined from the neutron β-decay data.

The r.h.s. of Eq. (1) is given by a sum of two series in powers of $1/Q^2$ (OPE HT corrections) and in powers of the QCD running coupling $\alpha_{\rm s}(Q^2)$ (pQCD radiative corrections). A theoretical analysis therefore includes the perturbative and non-perturbative components related to each other. Until very recently, the pQCD contribution to BSR was known [11] up to the third order $\sim \alpha_{\rm s}^3$. So far, the corresponding expression has been used in many studies, in particular, for extraction of the $\alpha_{\rm s}$ values at low momentum scales [12].

One of the actual theoretical subjects is the interplay between the HT and higher order pQCD corrections at low Q, which has recently been studied in Refs. [13] at the three-loop level. There, it was shown that the satisfactory description of the data down to $Q_{min} \sim \Lambda_{\rm QCD} \simeq 350$ MeV can be achieved within the analytic perturbation theory (APT), the ghost-free modification of pQCD.

The APT approach is based on the causality principle implemented as the analyticity imperative in the complex Q^2-plane for the QCD coupling $\alpha_{\rm s}(Q^2)$ in the form of the Källen-Lehmann spectral representation [14] and on the demand of compatibility with linear integral transformations [15] (for an overview on the APT concept and results, see Ref. [16]). It is well-known that in the APT, the theoretical ambiguity associated with pQCD higher-loop corrections is diminished (see Ref. [17]), and also results are practically renormalization scheme independent [18].

The first APT analysis for the Bjorken sum rule has been performed in [19] and has not been consider higher-twist effects because for that period the values of higher-twist parameters have not been well determined neither experimentally nor theoretically. The four-loop expression (Ref. [20]) for the pQCD contribution to the BSR gives us a reasonable motivation for a new analysis of the combined JLab data on $\Gamma_1^{p-n}(Q^2)$ at low $0.05 < Q^2 < 3.0\ \text{GeV}^2$ accounting for up to α_s^4-order in both the (standard) PT and APT approaches.

In this work we continue the researches begun in Ref. [13] of the four-loop ($\sim \alpha_s^4$) contribution to the BSR. We compare APT result with standard PT description and extract higher twist parameters for the BSR.

2 The perturbative QCD contribution

Commonly, one represents the Bjorken integral (1) as a sum of the perturbative and the HT contributions

$$\Gamma_1^{p-n}(Q^2) = \frac{g_A}{6}\left[1 - \Delta_{\rm Bj}(Q^2)\right] + \sum_{i=2}^{\infty} \frac{\mu_{2i}}{Q^{2i-2}}\,. \tag{2}$$

The perturbative QCD correction defined by the coefficient function $\Delta_{\rm Bj} = 1 - C_{\rm Bj}(\alpha_s)$ has a form of the power series in the QCD running coupling α_s. At the up-to-date four-loop level it looks like

$$\Delta_{\rm Bj}^{\rm PT}(Q^2) = \sum_{k\leq 4} c_k\, \bar{\alpha}_s{}^k(Q^2)\,. \tag{3}$$

Here, the bar symbol atop of running coupling α_s indicates that it is considered as a function of Q^2, rather than specific value of this function at some point; the numerical expansion coefficients c_i in the modified minimal subtraction ($\overline{\rm MS}$) scheme, for three active flavors, $n_f = 3$, read $c_1 = 1/\pi = 0.31831$, $c_2 = 0.36307$ [21], $c_3 = 0.65197$ [11] and $c_4 = 1.8042$ [20]. Besides, the four-loop running coupling $\bar{\alpha}_s$ is defined as a solution of the renormalization group (RG) equation

$$\frac{d\bar{\alpha}_s\,(Q^2)}{dL} = \beta(\bar{\alpha}_s)\,; \quad \beta(\alpha_s) = \sum_{0\leq k\leq 3} \beta_k\, \alpha_s{}^{k+2}\,, \tag{4}$$

where $L = \ln(Q^2/\Lambda^2)$ and β_k are the coefficients of the β-function. In the further analysis the exact solutions of the RG equation (4) in the $\overline{\text{MS}}$-scheme (see Ref. [22] for details) are used.

2.1 Analytic Perturbation Theory

The moments of the structure functions are analytic functions in the complex Q^2-plane with a cut along the negative part of the real axis (see, e.g., Ref. [23]). The perturbative representation (3) violates these analytic properties due to the unphysical singularities of $\bar{\alpha}_s(Q^2)$ in the physical region $Q^2 > 0$. To resolve the issue, we apply the APT method [14, 16], which allows one to combine the RG invariance with correct analytical properties of the QCD coupling and observables. In particular, the four-loop APT expression for the perturbative part of the BSR is given by the non-power functional expansion

$$\Delta_{\mathrm{Bj}}^{\mathrm{APT}}(Q^2) = \sum_{k\leq 4} c_k\, \mathcal{A}_k(Q^2)\,. \tag{5}$$

Here the coefficients c_k are the same as in Eq. (3), and the functions $\mathcal{A}_k(Q^2)$ are defined through the spectral functions $\varrho_k(\sigma) \equiv \mathrm{Im}\left[\alpha_s^k(-\sigma - i\epsilon)\right]$ by the spectral integral

$$\mathcal{A}_k(Q^2) = \frac{1}{\pi}\int\limits_0^\infty d\sigma\, \frac{\varrho_k(\sigma)}{\sigma + Q^2}\,. \tag{6}$$

At large momentum transfers, all the functions $\mathcal{A}_k(Q^2)$ become proportional to the k-th power of the usual perturbative coupling $[\alpha_s(Q^2)]^k$ and the expansion (5) reduces to the power series (3). However, at small enough $Q \leq 1-2$ GeV the properties of the non-power expansion (5) become considerably different from the PT one (3) (see, e.g., Ref. [18] for details).

2.2 The Q^2-dependence

Now we analyze the Q^2-dependence of the BSR within the both PT and APT approaches in different orders (NLO, N^{2}LO and N^{3}LO) of the perturbative expansions (3) and (5), respectively. As a normalization point, we take the most accurate α_s-value $\alpha_s(M_Z) = 0.1184 \pm 0.0007$ [10, 24]. To take into account flavor thresholds, we apply the matching conditions for the $\alpha_s(Q^2)$ which are rather nontrivial in higher PT orders (see Refs. [25]).

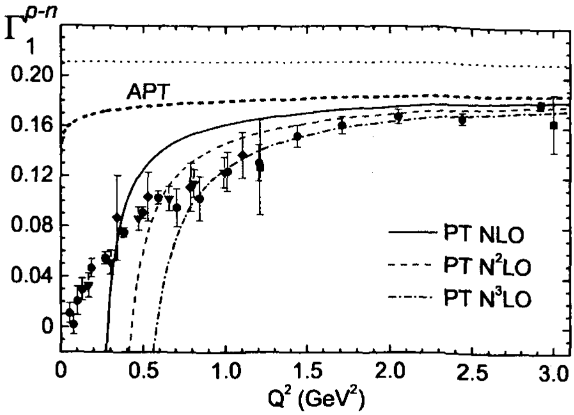

Figure 1. Perturbative part of the BSR as a function of Q^2 in different orders in both the APT and standard PT approaches against the combined set of the data.

Following to analysis in Ref. [26], our matched calculation for the four-loop $\overline{\mathrm{MS}}$-coupling gives $\Lambda^{(n_f=3)} = 336 \pm 10$ MeV. Note, we would obtain practically the same results, but with larger errors, if we choose the pseudo-observable value $R(M_Z^2) = 1.03904 \pm 0.00087$ as a starting point [27], which corresponds to the four-loop coupling value $\alpha_s(M_Z) = 0.1190 \pm 0.0026$.

In Fig. 1, we illustrate the behavior of the perturbative part of the BSR in different orders (NLO, N^{2}LO, N^{3}LO) in both the PT and APT approaches. The APT curves in all three orders practically (at about 1 % accuracy) coincide with each other, so we represent the APT result by a single heavy broken curve. For completeness, we also give the combined SLAC and JLab data on $\Gamma_1^{p-n}(Q^2)$ used in our analysis. The SLAC points [2, 3, 4, 5] are denoted by squares, the JLab CLAS Hall A 2002 – by downward pointing triangles, the JLab CLAS Hall B 2003 – by diamonds [28], and the most recent JLab data [9] – by circles. The horizontal dotted line shows the limiting value $\Gamma_1^{p-n}(\infty) = g_A/6$.

One sees from Fig. 1 that the PT four-loop curve describes the data quite well at $Q^2 \geq 0.7$ GeV^2. Besides, the three- and four-loop curves lie within the experimental errors (which is of the same order as both the contributions) and the data cannot distinguish between them.

At the same time, at $Q^2 \leq 0.7$ GeV^2 the four-loop curve describes the data equally bad as the three- and two-loop ones. This is a signal that one has to account for HT contributions, which strongly depend on the PT order used for its extraction [13](also in Table 1, below).

This changes in the APT case where the higher-loop stability is achieved due to the absence of unphysical singularities. At the same time, the deviation of APT curve from the data clearly shows for necessity of the HT contribution which is also quite stable [13].

This situation may be considered as a hint of the transition of PT series to the asymptotic regime (while APT series remains convergent) for $Q^2 \sim 0.7$ GeV^2. We explore this possibility in more detail.

2.3 Convergence of the PT and APT expansions

Clearly, at low Q^2 a value of the strong coupling is quite large,questioning the convergence of perturbative QCD series. The PT power series up to the known four-loop term (c.f. Eq. (3)) reads

$$\begin{aligned}\Delta_{\rm Bj}^{\rm PT}(\alpha_s) &= 0.3183\,\alpha_s + 0.3631\,\alpha_s^2 + \\ &+ 0.6520\,\alpha_s^3 + 1.804\,\alpha_s^4 = \sum_{i\le 4} \delta_i(\alpha_s), \qquad (7)\end{aligned}$$

where δ_i is the i-th term. The quantitative resemblance of the coefficients rise to the factorial growth $c_k \sim c_1\,(k-1)!$ is evident although for a definite statement one requires more deep analysis. This observation allows one to estimate the value $\alpha_s^* \sim 1/3$ as a critical one $(\delta_3(\alpha_s^*) \simeq \delta_4(\alpha_s^*))$.

To test that, we present in Fig. 2 the relative contributions of separate terms in the four-loop expansion (7)

$$\mathrm{N}_i(Q^2) = \delta_i(Q^2)/\Delta_{\rm Bj}(Q^2). \qquad (8)$$

As it is seen from Fig. 2, in the region $Q^2 < 1$ GeV^2 the dominant contribution to the pQCD correction, comes from the four-loop term $\sim \alpha_s^4$; its relative contribution increases with decreasing Q^2. This may be considered as an extra argument supporting an asymptotic character of the PT expansion in this region.

In the region $Q^2 > 2$ GeV^2 the situation is reverse – the major contribution comes from one- and two-loop orders there. Analogous curves for the APT series given by Eq. (5) are presented in Fig. 3.

Figures 2 and 3 demonstrate the essential difference between the PT and APT cases, namely, the APT expansion converges much better than

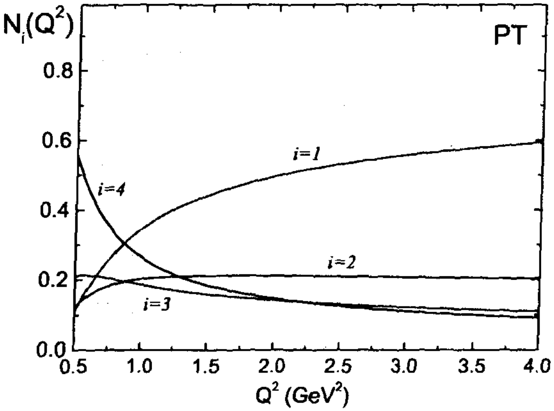

Figure 2. The Q^2-dependence of the relative contributions at the four-loop level in the PT approach. Four-loop PT order overshoots the three-loop one at $Q^2 \leq 2$ GeV^2, so it does not improve the accuracy of the PT prediction compared to the three-loop one.

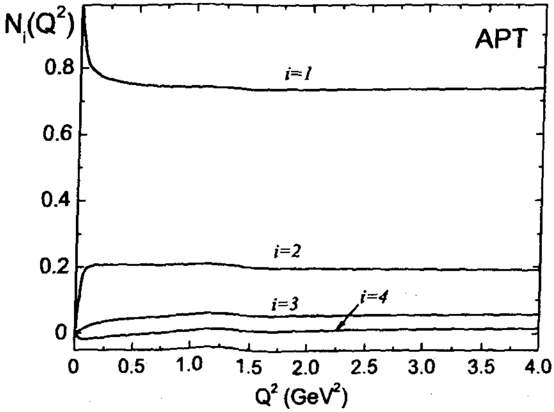

Figure 3. The Q^2-dependence of the relative contributions of the perturbative expansion terms in Eq. (5) in the APT approach.

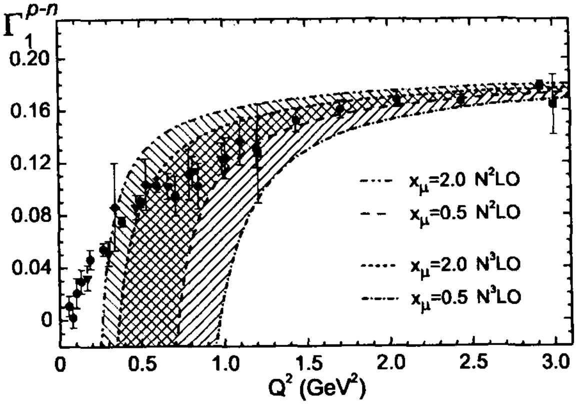

Figure 4. The μ-scale ambiguities for the perturbative part of the BSR versus Q^2 for three- (shaded region between dash-dot-dotted and dashed curves) and four-loop (shaded region between short-dashed and dash-dotted curves) orders of pQCD related to x_μ in the interval $0.5 \div 2$. These two regions have similar widths and only slightly shifted w.r.t. each other, so the differences between three- and four-loop results are within the data error bars.

the PT one. In the APT case, the higher order contributions are stable at all Q^2 values, with the one-loop contribution giving about 70 %, two-loop – 20 %, three-loop – not exceeds 5%, and four-loop – up to 1 %. The four-loop APT term can be important, only if the theoretical accuracy to better than 1 % will be actual.

2.4 The μ-scale dependence

As it is known, any observable obtained to all orders in pQCD expansion should be independent of the normalization scale μ, but in any truncated-order perturbative series the cancellation is not perfect, such that the pQCD predictions do depend on the μ-scale choice (for a review see, Ref. [24]).

In order to estimate this μ-dependence of Γ_1^{p-n} we use the four-loop expression for the coefficient function $C_{\rm Bj}$ [20], which can be rewritten as

$$C_{\rm Bj}\left(x_\mu=\frac{\mu^2}{Q^2},\alpha_{\rm s}\right)=1-0.3183\,\alpha_{\rm s}(\mu^2)-[0.3631+0.2280\ln x_\mu]\,\alpha_{\rm s}^2(\mu^2)$$
$$+\left[-0.6520-0.6491\ln x_\mu-0.16327\ln^2 x_\mu\right]\alpha_{\rm s}^3(\mu^2)$$
$$+\left[-1.804-1.798\ln x_\mu-0.7897\ln^2 x_\mu-0.1169\ln^3 x_\mu\right]\alpha_{\rm s}^4(\mu^2)\,.$$

Here, we introduced the dimensionless parameter x_μ ($\mu^2 = x_\mu Q^2$), which is changed within the interval $x_\mu = 0.5 \div 2$ (see, for example, the analysis in Ref. [27]). Further, we compare the μ-scale ambiguities for the three- and four-loop PT expressions.

In Fig. 4, the perturbative part of the BSR is plotted as a function of Q^2 in three- and four-loop PT orders corresponding to x_μ in the interval $0.5 \div 2$. The width of the arising strip for the four-loop expression is close to the one for the three-loop approximation in the highest JLab region $Q^2 \sim 3$ GeV^2 [1], so these approximations provide the description of the data with comparable accuracy, as it has been noticed above. Thus, in the low-energy domain the N^{3}LO approximation in the PT case does not improve the data description compared to the N^{2}LO one (see also Fig. 2).

For larger Q^2, indeed, four-loop approximation does improve the data description compared to the three-loop one (see, for example, extraction of $\alpha_{\rm s}$ from hadronic tau-decays [27]).

3 Higher twists contribution

Now, using expression (2) fitted to the above mentioned data [9, 28] we can extract the coefficients μ_{2i} of the HT OPE corrections. The chosen minimal borders of fitting domains in Q^2 allow to get $\chi^2_{d.f.} < 1$ with the monotonous behavior of the resulting fitted curves.

In Figs. 5 and 6 we present the results of 1- and 3-parametric fits in various orders of PT and APT. The corresponding fit results for HT terms, extracted in different orders of PT and APT, are given in Table 1 (all numerical results are normalized to the corresponding powers of the nucleon mass M). From these figures and Table follows that APT allows

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[1] One can find that an account for four-loop contribution leads to a decrease of the μ-dependence if $Q^2 \geq 5$ GeV^2 which is currently outside the JLab kinematical range, but will be accessible by JLab after the scheduled upgrade.

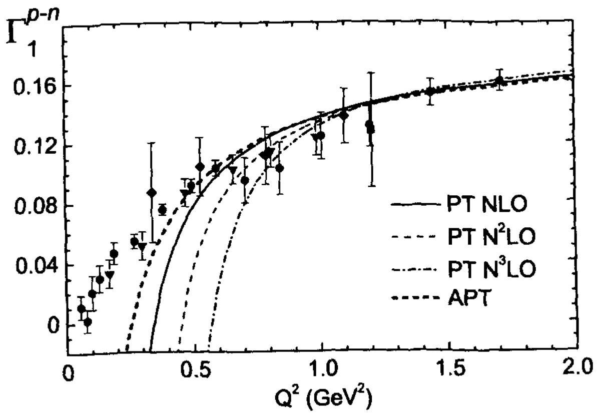

Figure 5. The one-parametric μ_4-fits of the BSR JLab data in various orders of the PT and the all-order APT expansions. In the PT case, the four-loop result does not improve the data description compared to the three-loop one. In the APT, even the NLO approximation turns out to be sufficient (see also Fig. 3).

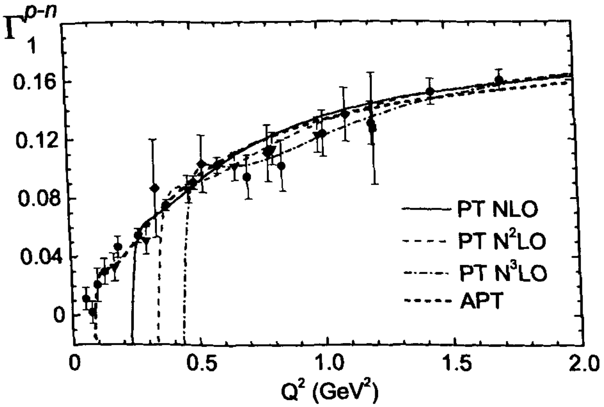

Figure 6. The three-parametric $\mu_{4,6,8}$-fits of the BSR JLab data in various orders of the PT and the all-order APT expansions.

Table 1. Results of HT extraction from the JLab data on the BSR in various orders of PT and all orders of APT with left border Q^2_{min} [GeV^2] of fitting domain.

Method	Q^2_{min}	μ_4/M^2	μ_6/M^4	μ_8/M^6	$\chi^2_{d.f.}$
The best μ_4-fit results					
PT NLO	0.5	−0.028(3)	–	–	0.80
PT N^{2}LO	0.66	−0.014(5)	–	–	0.59
PT N^{3}LO	0.71	0.006(7)	–	–	0.51
APT	0.47	−0.050(2)	–	–	0.82
The best $\mu_{4,6,8}$-fit results					
PT NLO	0.27	−0.026(9)	−0.01(1)	0.008(4)	0.69
PT N^{2}LO	0.34	0.01(2)	−0.06(4)	0.04(2)	0.67
PT N^{3}LO	0.47	0.05(3)	−0.17(9)	0.12(6)	0.46
APT	0.08	−0.061(2)	0.009(1)	−0.0004(1)	0.91

one to move further down to $Q^2 \sim 0.1$ GeV^2 in description of the data [13]. At the same time, in the framework of the standard PT the lower border shifts up to higher Q^2 scales with increasing of the PT expansion order. This is due to the more strong resulting singularities in the higher powers of usual strong coupling.

4 Sensitivity of the higher twists to Λ_{QCD} variations

In the above analysis, we normalized α_s at the Z-boson mass scale and then fixed the value of the Λ parameter separately in each order in α_s approximation (it was sufficient for understanding the role of the fourth order in the PT/APT series). However, the corresponding values of the Λ extracted in this way are slightly different from ones obtained from the direct QCD analysis of polarized deep inelastic scattering data (see, e.g., Ref. [29]) and also from the analysis in Ref. [30] where the "denominator formula" [31] was used. Having this in mind, we investigate additionally the sensitivity of the extracted values of the HT term μ_4 to the Λ in various orders of PT. In the APT, the sensitivity of μ_4 to the Λ is weak, and it

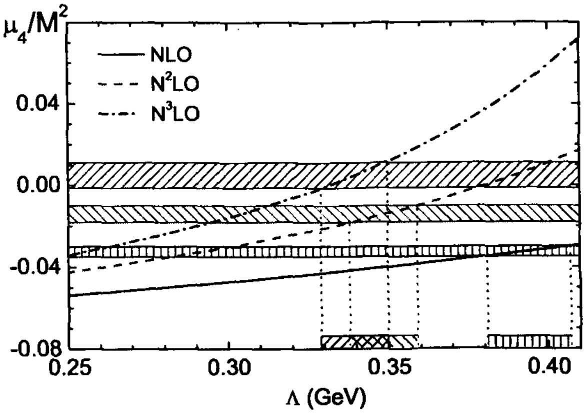

Figure 7. Value of the HT coefficient μ_4 extracted from the JLab data using the PT at different orders at $Q^2_{min} = 0.66$ GeV^2 with error bands. Vertical lines denote the corresponding uncertainty ranges in the Λ value. The ranges corresponding to N^{2}LO and N^{3}LO approximations have similar sizes and overlap with each other, so the four-loop result does not improve the stability w.r.t. Λ variations compared to the three-loop one.

does not depend on the order of the loop expansion. Correspondingly, the values of the HT coefficients turn out to be considerably more precise than those extracted in the PT approach (see also Table 1).

In Fig. 7 we show values of the coefficient μ_4 extracted from the JLab data using two-, three- and four-loop PT at $Q^2_{min} = 0.66$ GeV^2 vs the Λ variation. One can see that the PT does not lead to a stable results. The extracted coefficient μ_4 changes quite strongly between different orders of the PT expansion. And it happens in both in absolute value and sign, namely, at $\Lambda > 320$ MeV it becomes positive in the four-loop PT order. The sensitivity arises at higher PT orders.

On the other hand, these data tell us that the absolute value of μ_4 decreases with the order of PT and just at four-loop order becomes compatible to zero. This may be considered as a manifestation of duality between higher orders of PT and HT (see Ref. [32] and as well as [13, 33, 34, 35]). Moreover, when PT series reveals the asymptotic behavior (and becomes closer to data), the HT (which may be considered as a contribution completing the PT series) can be reduced to zero.

5 Summary and Conclusion

We performed the QCD analysis of the precise low energy JLab data on the BSR in the N^3LO PT order and extracted the higher twist terms using the four-loop expression for the QCD contribution to the Bjorken integral published recently in Ref. [20]. Our main observations are:

i) The four-loop PT approximation (without an account for HT) provides good description of the data for the highest JLab $Q^2 \sim 3$ GeV^2. For several data points there arises an impression that the four-loop description turns out to be better than the three-loop one. At the same time, the order of magnitude of both these contributions is the same as an experimental error, so a more precise statement can hardly be made.

ii) The magnitude of HT ($\mathcal{O}(1/Q^2)$) decreases (see Table 1) with an order of PT and becomes compatible to zero at the four-loop level.

iii) For lower $Q^2 \leq 0.7$ GeV^2 the four-loop PT term does not help to describe the data. Meanwhile, as it was shown earlier [13], the APT application leads to higher loops stability of the HT extraction. In turn, this results in accurate data description down to $Q^2 \sim 0.1$ GeV^2 always at the two-loop APT level (see Fig. 6).

Our first concluding impression is that all these features may indicate that the asymptotic nature of the QCD PT series is revealed at the four-loop level at $Q^2 \sim 1$ GeV^2. This conjecture is confirmed by the analysis of relative contributions of various PT terms, as well as by that of unphysical μ-dependence.

The second one concerns with puzzling stability of the APT-HT interplay As far as the difference between pQCD expansions (3), (7) and the APT one (5) is due to non-analytic in α_s, power-like in $1/Q^2$ terms (quite similar to the HT ones) this rises the hope of further insight into the structure of these non-analytic and power-like contributions into the observables.

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