Non-Perturbative Expansion Technique and Threshold Resummation for the Inclusive τ-decay and e^+e^- Annihilation into Hadrons Processes

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(Received 29 May 2001)

The method of non-perturbative a-expansion, which gives a self-consistent description of both spacelike and timelike regions, is applied to describe the D-functions corresponding to the inclusive τ-decay and e^+e^- annihilation into hadrons data. Thresholds effects are taken into account via a new relativistic Coulomb-like resummation factor. It is shown the method proposed leads to good agreement with experimental data down to the lowest energy scale.

Key words: non-perturbative a-expansion, hadrons processes, thresholds effects, relativistic Coulomb-like resummation, τ-decay, e^+e^- annihilation

PACS numbers: 13.25.H; 13.60; 13.25.F

1 Introduction

Specific feature of quantum field theory is that a sufficiently complete study of the structure of a quantum field model within the framework of perturbative approach is not enough, even in theories with a small coupling constant. Numerous publications are devoted to the problem of going beyond perturbation theory. However, many of them use model assumptions and phenomenological parameters which are not involved into the Lagrangian. Clearly, that it is desirable to use a theoretical method based on a minimal number of additional parameters and allowing a nonperturbative region to be considered. The theoretical method we will use is the nonperturbative expansion technique [1] based on the idea of variational perturbation theory (see [2] for a review) which, in the case of QCD, leads to a new small expansion parameter, a. Even going into the infrared region of small momenta where the running coupling becomes large and the standard perturbative expansion fails, the a-expansion parameter remains small and the approach holds valid [3].

In comparing theoretical predictions with experimental data, it is important to connect measured quantities with "simplest" theoretical objects to check direct consequences of the theory without using model assumptions in an essential manner. Some single-argument functions which are directly connected with experimentally measured quantities can play the role of these objects. A theoretical description of inclusive processes can be made in terms of functions of this sort. Let us mention, among them, the hadronic correlator $\Pi(s)$ and the corresponding Adler function [4], D, that appear in the process of e^+e^- annihilation into hadrons and the inclusive decay of the τ lepton.

The cross-section for e^+e^- annihilation into hadrons or its ratio to the leptonic cross-section, $R(s)$, have a resonance structure that is difficult to describe, at the present stage of a theory, without model considerations. Moreover, the basic method of calculations in quantum field theory, perturbation theory, becomes ill-defined due to the so-called threshold singularities. These problems can, in principle, be avoided if one considers a "smeared" quantity [5]

$$R_\Delta(s) = \frac{\Delta}{\pi} \int_0^\infty ds' \frac{R(s')}{(s-s')^2 + \Delta^2} \, . \qquad (1)$$

However, a straightforward usage of conventional perturbation theory to calculate R_Δ is not possible. Indeed, if the QCD contribution to the function $R(s)$ in Eq. (1) is, as usual, parametrized by the perturbative running coupling that has unphysical singularities, it is difficult to define the integral on the right-hand side. Moreover, the standard method of the renormalization group gives a Q^2-evolution law of the running coupling in the Euclidean region, and there is the question of how to parametrize a quantity, for example, $R(s)$, defined for timelike momentum transfers [6]. To perform this procedure self-consistently, it is important to maintain correct analytic properties of the hadronic correlator which are violated in perturbation theory. Within the nonperturbative a-expansion it is possible to maintain such analytic properties and to self-consistently determine the effective coupling in the Minkowskian region [7].[1]

Another function, which characterizes the process of e^+e^- annihilation into hadrons and can be extracted from experimental data, is the Adler function

$$D(Q^2) = -Q^2 \frac{d\Pi}{dQ^2} = Q^2 \int_0^\infty ds \frac{R(s)}{(s+Q^2)^2} \, . \qquad (2)$$

The D-function defined in the Euclidean region for a positive momentum Q^2 is a smooth function, and thus it is not necessary to apply any "smearing" procedure in order to be able to compare theoretical results with experimental data. An "experimental" curve for this function which is related to the process of e^+e^- annihilation into hadrons has been obtained in [10]. We will also consider the "light" D-function corresponding to the τ decay data.

2 Relativistic threshold factor

In the threshold region one cannot truncate the perturbative series. Threshold singularities of the Feynman diagrams of the form $(\alpha/v)^n$ have to be summarized. This resummation, performed on the basis of the nonrelativistic Schrödinger equation with the Coulomb potential $V(r) = -\alpha/r$, leads to the Sommerfeld-Sakharov factor [11, 12]

$$S_{\rm nr} = \frac{X_{\rm nr}}{1 - \exp(-X_{\rm nr})} \, , \qquad X_{\rm nr} = \frac{\pi\,\alpha}{v_{\rm nr}} \, , \qquad (3)$$

which is related to the wave function of the continuous spectrum at the origin, $|\psi(0)|^2$. Here $v_{\rm nr}$ is the velocity of the particle. An expansion of Eq. (3) in a power series in the coupling constant α reproduces the threshold singularities of the Feynman diagrams in the form $(\alpha/v)^n$.

A description of quark-antiquark systems near threshold also requires such type of resummation. The S-factor appears in the parametrization of the imaginary part of the quark current correlator, the Drell ratio $R(s)$, which can be approximated in terms of the Bethe-Salpeter (BS) amplitude of two charged particles $\chi_{\rm BS}(x)$ at $x = 0$ [13]. The nonrelativistic replacement of this amplitude by the wave function which obeys the Schrödinger equation with the Coulomb potential, leads to the approximation (3) with $\alpha \to 4\alpha_s/3$, for QCD.

In the relativistic theory the nonrelativistic approximation needs to be modified. To use the S-factor within such a relativistic regime one usually uses the simple substitution $v_{\rm nr} \to v$ with $v = \sqrt{1 - 4m^2/s}$. However, the corresponding relativistic generalization of the S-factor is obviously not unique, for there are numerous ways of expressing the nonrelativistic velocity in terms of the relativistic energy $\sqrt{s}$. For a systematic relativistic analysis of quark-antiquark systems, it is essential from the very beginning to have a relativistic generalization of the S-factor. Here we will use a new form for this relativistic factor proposed in [14].

The starting point of the consideration performed in [14] is the quasipotential (QP) approach proposed by Logunov and Tavkhelidze [15], in the form suggested by Kadyshevsky [16]. To find an explicit form for the relativistic S-factor one uses a transformation of the QP equation from momentum space into relativistic configuration space [17]. The local Coulomb potential defined in this representation

[1] The analytic approach to QCD [8] also leads to a well-defined procedure of analytic continuation [9].

has a QCD-like behavior in momentum space [18]. The possibility of using the QP approach to define the relativistic S-factor is based on the fact that the BS amplitude, which parameterizes the physical quantity $R(s)$, is taken at $x = 0$, therefore, in particular, at relative time $\tau = 0$. The QP wave function is defined as the BS amplitude at $\tau = 0$, and the R-ratio can be expressed through the QP wave function $\psi_{\rm QP}(\mathbf{p})$ by using the relation

$$\chi_{\rm BS}(x=0) = \int d\Omega_p \, \psi_{\rm QP}(\mathbf{p}) \,, \qquad (4)$$

where $d\Omega_p = (d\mathbf{p})/[(2\pi)^3 E_p]$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid [2] $E_p^2 - \mathbf{p}^2 = m^2$.

The QP equation in momentum space has the form

$$(2E - 2E_p)\, \psi(\mathbf{p}) = \int d\Omega_k \, V(\mathbf{p}(-)\mathbf{k})\, \psi(\mathbf{k}) \,. \qquad (5)$$

The proper Lorentz transformation, $\Lambda_\mathbf{k}$, means a translation in the Lobachevsky space

$$\begin{aligned} \Lambda_\mathbf{k}\, \mathbf{p} &\equiv \mathbf{p}(+)\mathbf{k} \\ &= \mathbf{p} + \mathbf{k}\left[\sqrt{1+\mathbf{p}^2} + \frac{\mathbf{p}\cdot\mathbf{k}}{1+\sqrt{1+\mathbf{k}^2}}\right] . \end{aligned} \qquad (6)$$

The role of the plane waves corresponding to the translations (6) is played by the following functions

$$\xi(\mathbf{p}, \mathbf{r}) = (E_p - \mathbf{p}\cdot\mathbf{n})^{-1-ir} \,, \qquad (7)$$

where $\mathbf{r} = \mathbf{n} r$ and $\mathbf{n}^2 = 1$. These functions correspond to the principal series of unitary representations of the Lorentz group and in the nonrelativistic limit ($p \ll 1$, $r \gg 1$) $\xi(\mathbf{p}, \mathbf{r}) \to \exp(i\mathbf{p}\cdot\mathbf{r})$. The orthogonality and completeness relations for these functions are

$$\begin{aligned} \int d\Omega_p \, \xi(\mathbf{p}, \mathbf{r})\, \xi^*(\mathbf{p}, \mathbf{r}') &= \delta(\mathbf{r} - \mathbf{r}') \,, \\ \int (d\mathbf{r})\, \xi(\mathbf{p}, \mathbf{r})\, \xi^*(\mathbf{k}, \mathbf{r}) &= (2\pi)^3 \delta(\mathbf{p}(-)\mathbf{k}), \end{aligned} \qquad (8)$$

where the relativistic momentum-space δ-function is $\delta(\mathbf{p}(-)\mathbf{k}) = \sqrt{1+\mathbf{p}^2}\, \delta(\mathbf{p} - \mathbf{k})$. The QP wave functions in the momentum and relativistic configuration representations are related as follows:

$$\begin{aligned} \psi(\mathbf{r}) &= \int d\Omega_p \, \xi(\mathbf{p}, \mathbf{r})\, \psi(\mathbf{p}) \,, \\ \psi(\mathbf{p}) &= \int (d\mathbf{r})\, \xi^*(\mathbf{p}, \mathbf{r})\, \psi(\mathbf{r}). \end{aligned} \qquad (9)$$

For a spherically symmetric potential the ξ-transform of Eq. (5) is the equation

$$\int d\Omega_p \, (d\mathbf{r}')\, (2E - 2E_p)\, \xi(\mathbf{p}, \mathbf{r}) \xi^*(\mathbf{p}, \mathbf{r}')\, \psi(\mathbf{r}') = V(r)\, \psi(\mathbf{r}), \qquad (10)$$

where the right hand side is local. Here the transform of the potential is given in terms of the same relativistic plane wave,

$$V(\mathbf{p}(-)\mathbf{k}) = \int (d\mathbf{r})\, \xi^*(\mathbf{p}(-)\mathbf{k}, \mathbf{r}) V(\mathbf{r}). \qquad (11)$$

The left hand side of this equation can be rewritten in a non-integral form by using the operator of the free Hamiltonian [17]

$$\begin{aligned} \hat{H}_0 = \cosh\left(i\frac{d}{dr}\right) &+ \frac{i}{r}\sinh\left(i\frac{d}{dr}\right) \\ &- \frac{\Delta_{\theta,\varphi}}{2r^2} \exp\left(i\frac{d}{dr}\right) , \end{aligned} \qquad (12)$$

where $\Delta_{\theta,\varphi}$ is the angular part of the Laplacian operator. The relation $\hat{H}_0 \xi(\mathbf{p}, \mathbf{r}) = E_p \xi(\mathbf{p}, \mathbf{r})$ allows one to re-express the equation in terms of finite differences

$$\left(2E - 2\hat{H}_0\right) \psi(\mathbf{r}) = V(r)\, \psi(\mathbf{r}) \,. \qquad (13)$$

Solutions of this equation, in principle, can contain arbitrary functions of r with period i, the so-called the i-periodic constants, which appear in the solutions due to the finite difference nature of the Hamiltonian (12). For some problems, such as defining the bound state spectrum, this i-periodic constant is not important. However, for the purpose of extracting the S-factor, one must develop a method

[2] In the following we will consider the case of two scalar particles with the same masses m and use the system of units $c = \hbar = m = 1$.

which avoids this ambiguity. For the Coulomb potential, Eq. (13) has been investigated in [19]. Other forms of the QP equation with the Coulomb potential has been considered in [20]. Our consideration based on [14].

Define the Coulomb potential in relativistic configuration space

$$V(r) = -\frac{\alpha}{r}\,. \qquad (14)$$

The ξ-transformation gives in momentum space the potential

$$V(\Delta) \sim \frac{1}{\chi_\Delta \sinh \chi_\Delta}\,, \qquad (15)$$

where the relative rapidity χ_Δ corresponds to $\mathbf{\Delta} = \mathbf{p}(-)\mathbf{k}$ and is defined in terms of the square of the momentum transfer by $Q^2 = -(p-k)^2 = 2(\cosh\chi_\Delta - 1)$. For large Q^2 the potential $V(\Delta)$ behaves as $(Q^2 \ln Q^2)^{-1}$, which reproduces the principal behavior of the QCD potential proportional to $\bar{\alpha}_s(Q^2)/Q^2$ with $\bar{\alpha}_s(Q^2)$ being the QCD running coupling [18].

According to Eqs. (4), (7), and (9), we find a relation between the required BS amplitude and the QP wave function, $\chi_{\rm BS}(x=0) = \psi_{\rm QP}(r=i)$. Performing a partial-wave analysis we further observe that the QP wave function for an ℓ-state will contain the generalized power $(-r)^{(\ell+1)} = i^{l+1}\Gamma(ir+l+1)/\Gamma(ir)$, which vanishes at $r=i$ for $\ell \neq 0$. Thus, we need only to consider the $\ell = 0$ wave function for which we can write $\psi(\mathbf{r}) = \psi(r)$. Introducing the function $R(r) = r\,\psi(r)$ into Eq. (10), we get

$$\frac{2}{\pi}\int_0^\infty d\chi' \int_0^\infty dr' \sin\chi' r \times \sin\chi' r' (2E - 2\cosh\chi') R(r') = V(r)\,R(r)\,. \qquad (16)$$

A solution of Eq. (16) with the Coulomb potential (14) one can seek in the form

$$R(r) = \int_\alpha^\beta d\zeta \, \exp(i\,r\,\zeta)\, R(\zeta)\,, \qquad (17)$$

where the ζ-integration is performed in the complex plane over a contour with endpoints (α, β) [21]. Substituting Eq. (17) into Eq. (16) one finds the equation

$$\int_\alpha^\beta d\zeta \, \exp(i\,r\,\zeta)\, (\,2E - 2\cosh\zeta)\; R(\zeta) = -\frac{\alpha}{r}\int_\alpha^\beta d\zeta\, \exp(i\,r\,\zeta)\, R(\zeta)\,, \qquad (18)$$

which, when we integrate by parts, yields the two equations

$$\exp(i\,r\,\zeta)\, (\,2E - 2\cosh\zeta)\; R(\zeta)\Big|_{\zeta=\alpha}^{\zeta=\beta} = 0 \qquad (19)$$

and

$$i\,\frac{d}{d\zeta}\Big[(\,2E - 2\cosh\zeta)\; R(\zeta)\Big] = -\,\alpha\, R(\zeta)\,. \qquad (20)$$

The solution of Eq. (20) is

$$R(\zeta) = C(\chi)\, \exp(\zeta)\Big[\exp(\zeta) - \exp(-\chi)\Big]^{A-1} \times \Big[\exp(\zeta) - \exp(\chi)\Big]^{-A-1}, \qquad (21)$$

where $A = i\alpha/(2\sinh\chi)$, $E = \cosh\chi$, and $C(\chi)$ is an arbitrary function of χ.

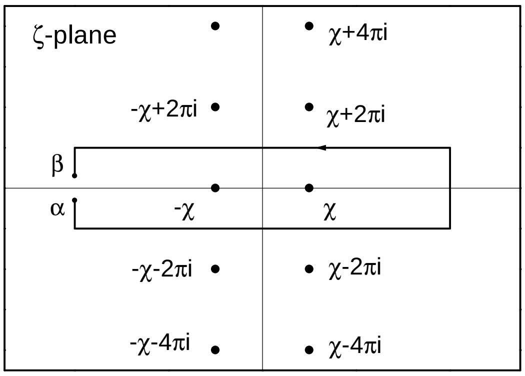

FIG. 1. Contour of integration in Eq. (17) and singularities of the function (21) in the complex ζ-plane.

The branch points of the function (21) are $\pm\chi + 2\pi in$ (see Fig. 1). The contour of integration must not intersect cuts which we take from $-\infty + 2\pi in$ to $\pm\chi + 2\pi in$. In the case when the interaction vanishes, $\alpha \to 0$, the solution $R(r)$ should reproduce the known free wave function $\sin\chi r/\sinh\chi$. Taking into account these remarks and Eq. (19) for the boundary values at (α,β), we take $\alpha=-R-i\varepsilon$, $\beta=-R+i\varepsilon$ with $R\to\infty$. The vertical part of the contour to the right is given by $\mathrm{Re}\,\zeta=+R$. It is also convenient for finding a connection to an integral representation of the hypergeometric function to take the horizontal parts of the contour to be characterized by $\mathrm{Im}\,\zeta=\pm\pi$ (see Fig. 1).

The resulting solution does not contain the i-periodic constant and reads[3]

$$R(r)=C(\chi)\sinh\pi r\int_{-\infty}^{\infty}dx\exp\left((ir+1)x\right)\times\frac{\Big[\exp(x)+\exp(-\chi)\Big]^{A-1}}{\Big[\exp(x)+\exp(\chi)\Big]^{A+1}}. \qquad (22)$$

Comparing the asymptotic form of Eq. (22) at $r\to\infty$ with the free wave function we can determine the constant $C(\chi)$ and calculate $|\psi_{\mathrm{QP}}(i)|^2$ which leads to the relativistic S-factor:

$$S(\chi)=\frac{X(\chi)}{1-\exp\left[-X(\chi)\right]},\qquad X(\chi)=\frac{\pi\alpha}{\sinh\chi}, \qquad (23)$$

where χ is the rapidity which related to s by $2\cosh\chi=\sqrt{s}$. The function $X(\chi)$ in Eq. (23) can be expressed in terms of v as $X(\chi)=\pi\alpha\sqrt{1-v^2}/v$.

3 Description of timelike and spacelike observables within the non-perturbative expansion method

For massless quarks, one can write down the timelike (Minkowskian) quantity $R(s)$ in the form

$$R(s)=3\sum_f q_f^2\left[1+r_0\lambda_s^{\mathrm{eff}}(s)\right], \qquad (24)$$

where the sum runs over quark flavors, q_f are quark charges and r_0 is the first perturbative coefficients that is renormalization-scheme independent. This expression includes the effective coupling defined in the Minkowskian region or, as we will say, in the s-channel, which is reflected in the subscript s. It should be stressed that, as it has been argued from general principles, the behavior of the effective couplings in the spacelike and the timelike domains cannot be symmetric [22].

[3]The representation of this solution in terms of the hypergeometric function can be found, for instance, by the substitution $x=\chi-\ln s$.

Within the a-expansion method the s-channel running coupling can be written as

$$\lambda_s^{(i)}(s)=\frac{1}{2\pi\mathrm{i}}\frac{1}{2\,\beta_0}\left[\,\phi^{(i)}(a_+)-\phi^{(i)}(a_-)\,\right], \qquad (25)$$

where $a_\pm$ obey the equation

$$F(a_\pm)=F(a_0)+\frac{2\,\beta_0}{C}\left(\ln\frac{s}{Q_0^2}\pm\mathrm{i}\,\pi\right). \qquad (26)$$

At the level $O(a^3)$, the function $\phi(a)$ has the form

$$\phi^{(3)}(a)=-4\ln a-\frac{72}{11}\frac{1}{1-a}+\frac{318}{121}\ln(1-a)+\frac{256}{363}\ln\left(1+\frac{9}{2}a\right). \qquad (27)$$

Similarly, a more complicated expression for the $O(a^5)$ level, we will use, can be derived.

The convenient way to incorporate quark mass effects is to use an approximate expression [5]

$$\tilde{R}(s)=3\sum_f q_f^2\,\Theta(s-4m_f^2)\,\mathcal{R}_f(s),\qquad \mathcal{R}_f(s)=T(v_f)\,\left[1+g(v_f)r_f(s)\right], \qquad (28)$$

where

$$T(v)=v\frac{3-v^2}{2},\qquad g(v)=\frac{4\pi}{3}\left[\frac{\pi}{2v}-\frac{3+v}{4}\left(\frac{\pi}{2}-\frac{3}{4\pi}\right)\right],\qquad v_f=\sqrt{1-\frac{4m_f^2}{s}}. \qquad (29)$$

The quantity $r_f(s)$ is defined by the s-channel effective coupling $\lambda_s^{\mathrm{eff}}(s)$. The smeared quantity (1) and the D-function (2) can be calculated by using (28) in the corresponding integrands. For MS-like renormalization schemes, one has to consider some matching procedure. To perform this matching procedure, we can require the s-channel running coupling and its derivative to be continuous functions in the vicinity of the threshold [7, 23].

To take into account the threshold resummation factor, we, following [24], modify the expression (28) by using the ansatz

$$\mathcal{R}(s) = T(v)\,\left[S(\chi) - \frac{1}{2}X(\chi) + g(v)\,r(s)\right]\,. \quad (30)$$

As the mass $m \to 0$, this expression leads to Eq. (24). We will use Eq. (30) in our analysis.

The non-strange vector contribution for the inclusive τ-lepton decay can be described in analogy with the e^+e^- annihilation into hadrons process. Using the theoretical expression for R_τ-ratio [25]

$$R_\tau^V = R^{(0)} \int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right)\mathcal{R}(s)\,, \quad (31)$$

where $R^{(0)}$ corresponds to the parton level, and measured value $R_\tau^V = 1.775 \pm 0.017$ [26], as an input, we extracted the value of parameter a_0 in Eq. (26) at the τ mass scale, $Q_0 = M_\tau$.

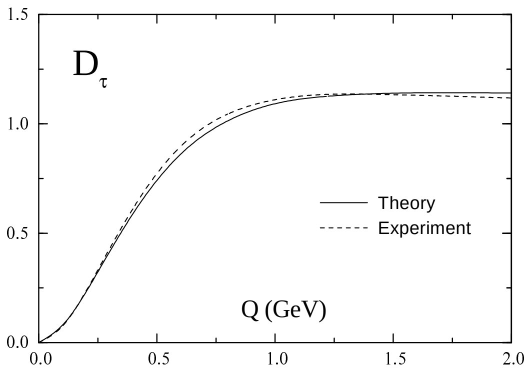

FIG. 2. The plot of the "light" D-function. The experimental curve corresponding to ALEPH data is taken from [27].

The "light" D-function with three active quarks is shown in Fig. 2, where we draw the experimental curve, as dashed line, which was extracted in [27] from the ALEPH data, and our theoretical result (solid line) obtained by using the following effective masses of light quarks $m_u = m_d = 260\,\mathrm{MeV}$ and $m_s = 400\,\mathrm{MeV}$. Virtually, the same values were used in [28, 29, 30, 31] to describe the region of low lying mesons. These values are close to the constituent quark masses and incorporate some non-perturbative effects. The shape of the infrared tail of the D-function is sensitive to the value of these masses.

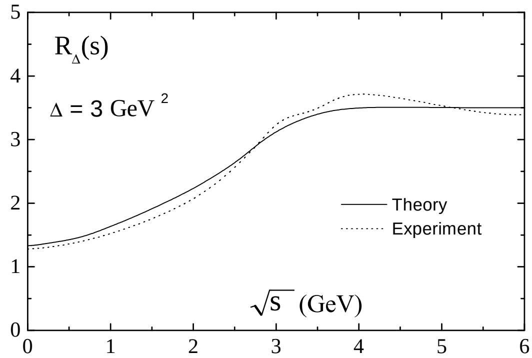

FIG. 3. The smeared quantity $R_\Delta(s)$ for $\Delta = 3$ GeV^2. The solid curve is our result. The smeared experimental curve is taken from protect[32].

In Fig. 3, we have presented the smeared function $R_\Delta(s)$ for $\Delta = 3$ GeV^2. We use the same masses for the light quarks as before and the following masses for heavy quarks $m_c = 1.3\,\mathrm{GeV}$ and $m_b = 4.7\,\mathrm{GeV}$. The smeared $R_\Delta(s)$ function for $\Delta \simeq$ 1–3 GeV^2 is less sensitive to the value of light quark masses as compared with the infrared tail of the D-function. The result for the D-function of the e^+e^--annihilation process which includes both the light and heavy quarks is plotted in Fig. 4.

4 Conclusions

We have considered the Minkowskian and Euclidean physical quantities obtained from the e^+e^- annihilation and τ decay experimental data. The method we used here is the non-perturbative approach based on an idea of variational perturbation theory which combines an optimization procedure of variational

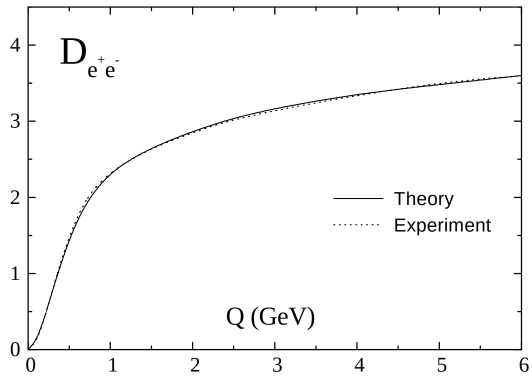

FIG. 4. The D-function for the process of e^+e^- annihilation into hadrons. The solid curve is our result for five active quarks. The experimental curve is taken from [10].

type with a regular method of calculating corrections. In the case of QCD the non-perturbative expansion parameter, a, obeys an equation whose solutions are always smaller than unity for any value of the original coupling constant. An important feature of this approach is the fact that for sufficiently small value of the running coupling the a-expansion reproduces the standard perturbative expansion, and, therefore, the perturbative high--energy physics is preserved. In moving to low energies, where ordinary perturbation theory breaks down ($\bar{\alpha}_s \simeq 1$), the parameter a remains small and we still stay within the region of applicability of the a-expansion method.

We have also used the new form of the threshold resummation factor. This relativistic factor could have a significant impact in interpreting strong-interaction physics. In many physically interesting cases, $R(s)$ occurs as a factor in an integrand, as, for example, for the case of inclusive τ decay, for smearing quantities, and for the Adler D function. Here the behavior of S at intermediate values of v becomes important.

In the nonrelativistic limit, $v \ll 1$, the relativistic S-factor (23) reproduces the nonrelativistic result (3). In the ultrarelativistic limit, as it has been argued in [33], the bound state spectrum vanishes as $m \to 0$ because the particle mass is the only dimensional parameter. This feature reflects an essential difference between potential models and quantum field theory, where an additional dimensional parameter appears. One can conclude that within a potential model, the S-factor which corresponds to the continuous spectrum should go to unity in the limit $m \to 0$. Thus, the relativistic resummation factor S obtained here reproduces both the expected nonrelativistic and ultrarelativistic limits and corresponds to a QCD-like Coulomb potential.

The experimental D-function turned out to be a smooth function without traces of the resonance structure of $R(s)$. One can expect that this object more precisely reflects the quark-hadron duality and is convenient for comparing theoretical predictions with experimental data. Note here that any finite order of the operator product expansion fails to describe the infrared tail of the D-function. Within the framework of nonperturbative a-expansion technique with the relativistic threshold factor, we have obtained a good agreement between our results and the experimental data down to the lowest energy scale both for Minkowskian and Euclidean quantities.

The authors would like to thank Professors K.A. Milton and A.N. Sissakian for cooperation and valuable discussions. Partial support of the work by the RFBR, grants 99-01-00091 and 99-02-17727, is gratefully acknowledged.

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