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Inclusive τ-Decay and QCD Sum Rules in Variational Perturbation Theory

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Nonperturbative expansion in quantum chromodynamics is applied to analyze low energy scale hadronic decay of τ-lepton data. Threshold singularities are summarized by a relativistic resummation S-factor. It is demonstrated that the suggested method allows good description of experimental D-function down to the lowest energy scale as well as parameters of ρ-meson.

Key words: quantum chromodynamics, nonperturbative methods, inclusive τ-decay, QCD sum rules

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1 Introduction

Data on τ-lepton decay into hadrons obtained with a record accuracy for hadronic processes [1, 2] give unique possibility for testing QCD at low energy scale.

In comparing theoretical predictions with experimental data, it is important to connect measured quantities with "simplest" theoretical objects to check direct consequences of the theory without using model assumptions in an essential manner. Some single-argument functions which are directly connected with experimentally measured quantities can play the role of these objects. Let us mention, among functions of this sort, the correlator $\Pi(q^2)$, that appears in the process of e^+e^- annihilation into hadrons and the inclusive decay of the τ-lepton into hadrons, and the corresponding Adler function, $D(Q^2)$ [3]. The last functions defined in the Euclidean region are smooth functions, and thus it is not necessary to apply any "smearing" procedure in order to be able to compare theoretical results with experimental data. The "experimental" curve for the Adler function corresponding to e^+e^- data and a comparative analysis has been presented in [4]. Here, we investigate the "light" D-function corresponding to inclusive τ decay data.

The basic method for performing calculations in quantum field theory is a perturbation theory. Its use along with the renormalization procedure allows important results to be obtained in quantum electrodynamics, in the theory of electroweak interactions, and in description of the perturbative region of QCD. However, a lot of problems of QCD require nonperturbative approaches. Perturbative approximation in QCD as a rule cannot be exhaustive in low energy region of a few GeV and nonperturbative contribution has to be included.

The development of nonperturbative methods has received a great deal of attention. The theoretical method we will use is the nonperturbative expansion technique [5] based on the idea of variational perturbation theory [6] which, in the case of QCD, leads to a new small expansion parameter. Even going into infrared region of small momenta where the running coupling becomes large and the standard perturbative expansion fails, the nonperturbative expansion parameter remains small and the approach holds valid [7].

2 Nonperturbative VPT-expansion

The method of variational perturbation theory (VPT) leads to the following connection between the expansion parameter a and the original coupling constant [5]

$$\lambda = \frac{\alpha_s}{4\pi} = \frac{1}{C}\,\frac{a^2}{(1-a)^3}. \tag{1}$$

For all values of the coupling constant $\lambda \geq 0$ the expansion parameter a obeys the inequality

$$0 \leq a < 1. \tag{2}$$

The positive parameter C plays the role of a variational parameter, which can be found by using an additional information [5, 6].

The Q^2-evolution of the expansion parameter a is defined by the renormalization group equation,

$$\ln\frac{Q^2}{Q_0^2} = \frac{C}{2\,\beta_0}\,\left[f(a) - f(a_0)\right]\,, \tag{3}$$

where $a_0 = a(Q_0^2)$, $\beta_0 = 11 - 2/3\,N_f$ is the one-loop coefficient of the renormalization group β-function in perturbative expansion, and N_f is the number of active quarks.

In this paper, as well as in [8, 9, 10], we use for our calculations the level $O(a^5)$. In this case the function $f(a)$ in (3) has the form

$$f^{(5)}(a) = \frac{1}{5(5+3B)}\sum_{i=1}^{3} x_i\,J(a,b_i) \tag{4}$$

with $B = \beta_1/(2C\beta_0)$, where two-loop coefficient $\beta_1 = 102 - 38N_f/3$, and

$$J(a,b) = -\frac{2}{a^2 b} - \frac{4}{ab^2} - \frac{9}{(1-a)(1-b)} - \frac{12}{ab} + \frac{4+12b+21b^2}{b^3}\ln a + \frac{30-21b}{(1-b)^2}\ln(1-a) - \frac{(2+b)^2}{b^3(1-b)^2}\ln(a-b)$$

with

$$x_i = \frac{1}{(b_i-b_j)(b_i-b_k)}\,. \tag{5}$$

Here indices $\{ijk\}$ are $\{123\}$ and cyclic permutations. The values of b_i are the roots of the equation $\psi(b_i) = 0$, where the function $\psi(a)$ is related to the β-function and reads

$$\psi(a) = 1 + \frac{9}{2}a + 2(6+a)a^2 + 5(5+3\,B)a^3\,. \tag{6}$$

For any values of Q^2, equation (3) has a unique solution $a(Q^2)$: $0 < a(Q^2) < 1$. Independently of a value a_0, an infrared limiting value is $a(0) = 1$.

The method of the renormalization group gives a Q^2-evolution law of the running coupling in the Euclidean region, and there is the question of how to parameterize a quantity, for example,

$$R(s) = \frac{1}{\pi}\mathrm{Im}\Pi(s+i0), \tag{7}$$

defined for timelike momentum transfers. To perform this procedure self-consistently, it is important to maintain correct analytic properties of the correlator $\Pi(q^2)$ which are violated in perturbation theory (PT). Within the framework of the analytic approach [11] and the nonperturbative VPT method used here, it is possible to maintain such analytic properties and self-consistently determine the effective coupling in the Minkowskian region [12, 13, 14].

3 τ-decay

The ratio of hadronic to leptonic widths for the inclusive decay of τ-lepton, R_τ, is the most precise one for extracting of values of the fundamental QCD parameters at a low energy scale. The initial theoretical expression for R_τ contains an integral over timelike momentum

$$R_\tau = 2\int_0^{M_\tau^2}\frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2 \times\left(1+2\frac{s}{M_\tau^2}\right)R(s) \tag{8}$$

which extends down to small s.

R_τ-ratio and the Adler function

$$D(Q^2) = Q^2\int_0^\infty ds\frac{R(s)}{(s+Q^2)^2} \tag{9}$$

cannot be directly calculated in the framework of standard PT. Indeed, the function $R(s)$ is parametrized by the perturbative running coupling which has nonphysical singularities (the ghost pole and corresponding cuts) and, therefore, is ill-defined in the region of small momenta.

Besides, the region of integration in (8) and (9) includes the vicinity of the quark-antiquark threshold. A description of quark-antiquark systems near threshold requires one to take into account the resummation factor which summarizes the threshold singularities of the perturbative series of the $(\alpha_s/v)^n$ type. In nonrelativistic approximation, this is the well known Sommerfeld-Sakharov factor [15, 16]. The corresponding relativistic resummation factor $S(\chi)$ (χ is the rapidity which related to s by $2m\cosh\chi = \sqrt{s}$) has been found in [17]. This relativistic resummation factor reproduces both expected nonrelativistic and ultrarelativistic limits and corresponds to a QCD-like quark-antiquark potential.

A convenient way to incorporate quark mass effects is to use an approximate expression [18, 19] which can be written as

$$R(s) = T(v)\;[1 + g(v)r(s)]\,\Theta(s-4m^2)\,, \quad (10)$$

where $r(s)$ is the massless QCD correction,

$$T(v) = v\frac{3-v^2}{2}\,, \quad (11)$$
$$g(v) = \frac{4\pi}{3}\left[\frac{\pi}{2v} - \frac{3+v}{4}\left(\frac{\pi}{2}-\frac{3}{4\pi}\right)\right],$$
$$v = \sqrt{1-\frac{4m^2}{s}}\,.$$

Effective quark mass, m, which incorporates some nonperturbative contributions, turns out to be close to the constituent masses. In the description of nonstrange vector component of the D-function we neglect the difference of the quark mass values and set $m_u = m_d = m$.

We represent $R(s)$ in the form

$$R(s) = R_0(s) + R_1(s)\,, \quad (12)$$

where

$$R_0(s) = T(v)\,S(\chi)\,, \quad (13)$$
$$R_1(s) = T(v)\left[\frac{\alpha_s(s)}{\pi}g(v) - \frac{1}{2}X(\chi)\right]\,.$$

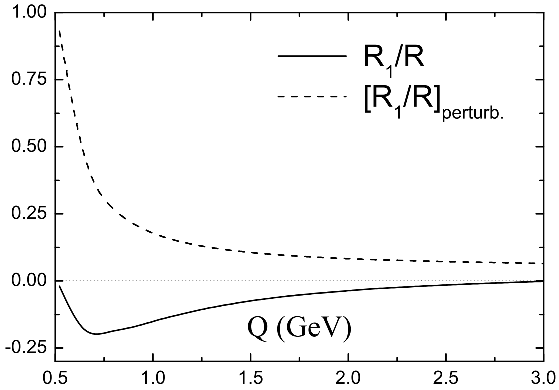

FIG. 1. Relative contributions to R.

In Fig. 1 we show relative contribution of the correction R_1 defined in (13) and R_1^{PT} defined in PT.

The "light" D-function corresponding to $N_f = 3$ quark world is plotted in Fig. 2. The τ-data for the vector spectral function of the ALEPH collaboration [1] have been used in [20] to extract the D_τ-function which we show as the dashed line in Fig. 2.

Our result was obtained by using the value of the quark masses $m = 260\,\text{MeV}$. Note, the similar value of the light quark mass has been received in the framework of analytic perturbation theory [21] and by using a covariant quark model with instanton-like quark-quark interaction [22, 23]. The shape of the infrared tail of the D-function is rather sensitive to the value of these masses.

We obtain the value

$$R_{\tau,V} = 1.77 \quad (14)$$

in the vector channel which agrees well with the experimental data presented by the ALEPH [1] and OPAL [2] collaborations

$$R_{\tau,V}^{\mathrm{ALEPH}} = 1.775 \pm 0.017, \quad (15)$$
$$R_{\tau,V}^{\mathrm{OPAL}} = 1.764 \pm 0.016. \quad (16)$$

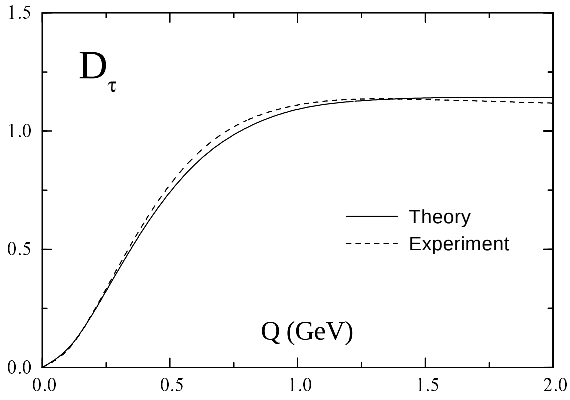

FIG. 2. The "light" D-function.

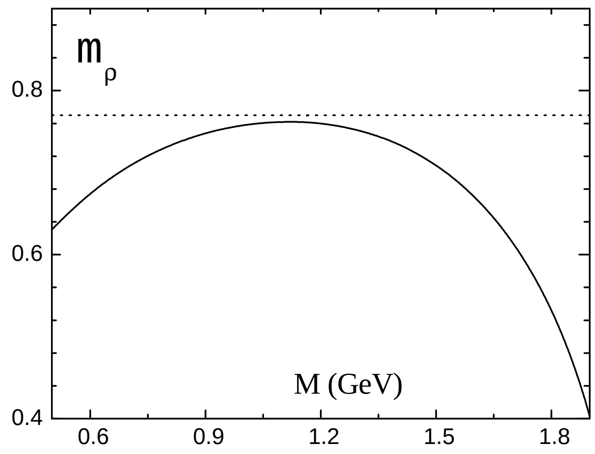

FIG. 3. Behavior of $m_\rho(M)$ defined by Eq. (18). Horizontal line corresponds to the experimental value.

4 ρ-meson parameters

To consider ρ-meson characteristics we use the QCD sum rules [24] with the following hadronic model for $R(s)$:

$$R^h(s) = \frac{2\pi}{g_\rho^2}\, m_\rho^2\, \delta(s - m_\rho^2) + \left(1 + \frac{\alpha_0}{\pi}\right) \theta(s - s_0)\,, \tag{17}$$

where the parameters $\alpha_0 \simeq 0.45$ and $s_0 \simeq 1.5$ GeV^2 [24, 25]. The first term in (17) corresponds to the resonance contribution (in narrow resonance approximation), the second term describes continuum contribution. Note, that by using experimental values of the ρ^0-parameters m_ρ and g_ρ^2, we reproduce well the "experimental" curve for the D-function represented in Fig. 2.

For the ρ-mass the Borel sum rules give

$$m_\rho^2 = \frac{1}{M_0(M^2, s_0)} \left[\mathfrak{M}_1(M^2) - \left(1 + \frac{\alpha_0}{\pi}\right) \times M^4 \left(1 + \frac{s_0}{M^2}\right) \exp\left(-s_0/M^2\right)\right]. \tag{18}$$

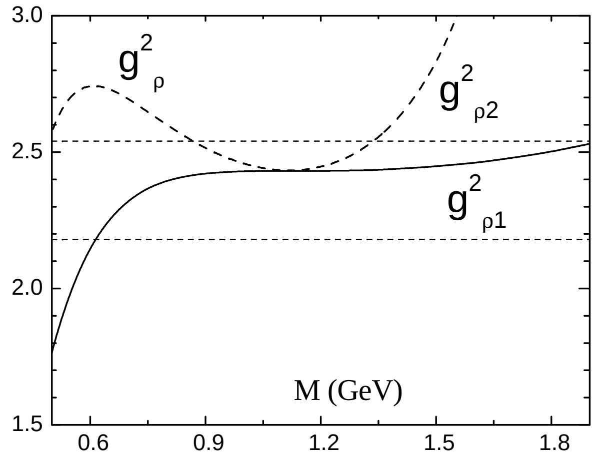

FIG. 4. Behavior of $g_\rho^2(M)$ defined by Eqs. (19) and (20). Horizontal lines correspond to the experimental "corridor".

For the width we can get two expressions

$$g_{\rho 1}^2 = \frac{2\pi\, m_\rho^2}{M_0(M^2, s_0)} \exp\left(-\frac{m_\rho^2}{M^2}\right), \tag{19}$$

$$g_{\rho 2}^2 = \frac{2\pi\, m_\rho^4}{M_1(M^2, s_0)} \exp\left(-\frac{m_\rho^2}{M^2}\right), \tag{20}$$

where

$$\mathfrak{M}_k(M^2) = \int\limits_0^\infty ds\, s^k \exp\left(-\frac{s}{M^2}\right) R(s) \qquad (21)$$

and

$$M_0(M^2, s_0) = \mathfrak{M}_0(M^2) - \left(1+\frac{\alpha_0}{\pi}\right) M^2 \exp\left(-\frac{s_0}{M^2}\right), \qquad (22)$$

$$M_1(M^2, s_0) = \mathfrak{M}_1(M^2) - \left(1+\frac{\alpha_0}{\pi}\right) \times M^2(s_0+M^2) \exp\left(-\frac{s_0}{M^2}\right). \qquad (23)$$

Results coming from Eqs. (18), (19) and (20) are shown in Figs. 3 and 4. Regions of stability give $m_\rho = 763$ MeV and $g_\rho^2 = 2.43$ which agree well with experimental data.

5 Conclusions

The method we used here is the non-perturbative approach based on the idea of variational perturbation theory which combines an optimization procedure of variational type with a regular method of calculating corrections. In the case of QCD the non-perturbative expansion parameter, a, obeys an equation whose solutions are always smaller than unity for any value of the original coupling constant. An important feature of this approach is the fact that for sufficiently small value of the running coupling the a-expansion reproduces the standard perturbative expansion, and, therefore, the perturbative high-energy physics is preserved. In moving to low energies, where ordinary perturbation theory breaks down ($\bar{\alpha}_s \simeq 1$), the parameter a remains small and we still stay within the region of applicability of the a-expansion method.

We have also used the new form of the threshold resummation S-factor. This relativistic factor could have a significant impact in interpreting of strong-interaction physics. In many physically interesting cases, $R(s)$ occurs as a factor in an integrand, as, for example, for the case of inclusive τ decay, for smearing quantities, and for the D-function. Here the behavior of S at intermediate values of v becomes important.

The experimental D-function turned out to be a smooth function without visible traces of the resonance structure of $R(s)$. One can expect that this object reflects more precisely the quark-hadron duality and is convenient for comparing theoretical predictions with experimental data. Note here that any finite order of the operator product expansion fails to describe the infrared tail of the D-function [20, 21]. Within the framework of nonperturbative a-expansion method with the relativistic threshold factor, we have obtained good agreement between theoretical results and experimental data down to the lowest energy scale.

The analysis performed leads to good agreement of theoretical results with experimental τ-data and ρ-meson parameters for the quark masses which are constant and close to the values of the constituent quark masses. More systematic consideration has to take into account the "dynamical" nature of quark mass as, for instance, in [26, 27], and use a mass function as, for instance, coming from the Dyson-Schwinger equations [28]. It is interesting that in these cases we come to similar results which we have received for the constant masses. Thus, the constant quark mass of order the constituent mass is useful model "average" object which can be used in simple analysis instead of more complicated consideration. Justification of this statement can also be obtained from the analysis of the QCD sum rules.

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