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ADVANTAGES OF APT IN QCD STUDY OF HADRONIC TAU DECAYS

O. P. Solovtsova[a]

Gomel State Technical University, Gomel, Belarus

A comparative analysis of different forms of approximations, which are applied to the description of the hadronic decays of the tau lepton, is given. Advantages and self-consistency of the method called analytic perturbation theory (APT) are demonstrated. It is shown that the use of the APT leads to the good description of certain inclusive functions associated with vector and axial-vector non-strange quark currents down to the lowest energy scale.

1 Introduction

The experimental data on the τ lepton decay into hadrons obtained with a record accuracy for hadronic processes [1–3] give a unique possibility for testing QCD at low energy scale. The τ lepton is the only lepton known at present whose mass, $M_\tau = 1.777\,\mathrm{GeV}$, is large enough in order to produce decays with a hadronic mode. At the same time, in the context of QCD, the mass is sufficiently small to allow one to investigate perturbative and non-perturbative QCD effects. The theoretical analysis of the hadronic decays of a heavy lepton was performed Tsai [4] before the experimental discovery of the τ lepton in 1975 and since then this process is intensively studied.

It is known, that perturbation theory (PT), which is a basic tool of calculations in quantum field theory, as a rule cannot be exhaustive in the low energy region of QCD. However, a structure of an initial perturbative approximation of some quantity is not a rigid construction fixed once and for all, but admits a considerable modification due to specific properties of quantum field theory. Such modification is based on further information of a general character about the sum of the series. In particular, the properties of renormalization-group (RG) invariance [5], which is lost in a finite order of the initial expansion, allow rearrangements of the perturbative series in terms of the invariant charge. In this case, the properties of the series change essentially. In distinction to the initial expression containing large logarithms, the expansion obtained within the RG method can be used for analyzing the ultraviolet region. However, the perturbative series so derived are ill-defined in the infrared region and the correct analytic properties of the series in the complex Q^2-plane are violated due to unphysical singularities of the perturbative running coupling, a ghost pole in the one-loop approximation (see discussion in [6, 7]). The difficulty associated with these unphysical singularities is overcome in the analytic approach proposed by Shirkov and Solovtsov [8]. This approach modifies the perturbative expansion on the basis of general properties of the theory so that the new approximations reflect fundamental principles of the theory— renormalization invariance, spectrality, and causality. In the new expansion the correct analytic properties are restored, and the property of RG invariance is preserved [8]. Further developments and applications of the Shirkov–Solovtsov analytic approach have been considered in many papers (see [9] as the recent review).

The original theoretical expression for the width $\Gamma(\tau^- \to hadrons\ \nu_\tau)$ involves integration over small values of timelike momentum [4]. The perturbative description with the standard running coupling becomes ill-defined in this region and some additional ansatz has to be applied to get a finite result for the hadronic width. To this end, one usually transforms the initial expression, by using Cauchy's theorem, to a contour representation for R_τ [10], which allows one to give meaning to the initial expression and, in principle, perform calculations in the framework of perturbative QCD. Assuming the validity of this transformation it is possible to present results in the form of a truncated power series with $\alpha_s(M_\tau)$ as the expansion parameter [11, 12]. There are also other approaches to evaluating the contour integral. The Le Diberder and Pich prescription [13] allows one to improve the convergence properties of the approximate series and reduce the renormalization scheme (RS) dependence of theoretical predictions. The possibility of using different approaches in the perturbative description of τ lepton decay leads to an uncertainty in the value of $\alpha_s(M_\tau)$ extracted from the experimental data. Moreover, any perturbative description is based on this contour representation, *i.e.*, on the possibility of converting the initial expression involving integration over timelike momenta into a contour integral in the complex momentum plane. To carry out this transition by using Cauchy's theorem requires certain analytic properties of the hadronic correlator or of the corresponding Adler function. However, the occurrence of incorrect analytic properties in

e-mail: [a]solovtsova@gstu.gomel.by

the conventional perturbative approximation makes it impossible to exploit Cauchy's theorem in this manner and, therefore, prevents rewriting the initial expression for R_τ in the form of a contour integral in the complex momentum-plane.

The method based on the Shirkov–Solovtsov analytic approach and called analytic perturbation theory (APT) [14] ensures the correct analytic properties of such important objects as the hadronic correlator or of the corresponding Adler function, leads to equality between the initial theoretical expression for the width $\Gamma(\tau^- \to \mathit{hadrons}\ \nu_\tau)$ and the corresponding contour representation.

The aim of this paper is to reveal features of the application of PT and APT expansions in studying the process of τ decay into hadrons, and in application the APT method for the description of hadronic widths associated with vector and axial-vector non-strange quark currents, and also for Adler functions, which are connected to these currents, down to the lowest energy scale.

2 Analytic perturbation theory

A main object in a description of hadronic decays of the τ lepton and of many other physical processes is the correlator $\Pi(q^2)$ or the corresponding Adler function $D(Q^2)$, which is connected to the correlator by the formula

$$D(Q^2) = -Q^2 \frac{d\Pi(-Q^2)}{dQ^2}\,. \tag{1}$$

We use the standard convention $Q^2 = -q^2 > 0$ in the Euclidean region.

The integral representation for the D-function is given in terms of the function $R(s) \equiv \mathrm{Im}\,\Pi(s)/\pi$:

$$D(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2}\, R(s)\,. \tag{2}$$

The representation (2) defines the function $D(Q^2)$ as the analytic function in the complex Q^2-plane with the cut along the negative real axis.

It is convenient to separate QCD contributions, $d(Q^2)$ and $r(s)$, in the functions $D \propto 1 + d$ and $R \propto 1 + r$, respectively, which are related by the formulae

$$d(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2} r(s), \tag{3}$$

$$r(s) = -\frac{1}{2\pi i} \int_{s-i\epsilon}^{s+i\epsilon} \frac{dz}{z} d(-z)\,. \tag{4}$$

The integration contour in (4) lies in the region of analyticity of the integrand and encircles the cut of $d(-z)$ on the positive real z axis.

In the APT the basic object is a spectral function $\rho(\sigma)$ which enters into some integral representation. In particular, for two-point functions, it is the Källén–Lehmann representation; whereas for structure functions for inelastic lepton–hadron scattering, the integral representation is that of Jost–Lehmann–Dyson. The spectral function $\rho(\sigma)$ for the objects under consideration here can be obtained by using the perturbative series as a initial approach. Truncated at the three-loop level, the perturbative d-function, rewritten in terms of the perturbative running coupling, $a_{\rm pt}(Q^2) = \bar{\alpha}_s(Q^2)/\pi$, is

$$d_{\rm pt}(Q^2) = a_{\rm pt}(Q^2) + d_1 a^2_{\rm pt}(Q^2) + d_2 a^3_{\rm pt}(Q^2)\,, \tag{5}$$

where in the $\overline{\rm MS}$ scheme for three active quarks ($n_f = 3$) relevant in τ decay, the expansion coefficients are $d_1^{\overline{\rm MS}} = 1.6398$ and $d_2^{\overline{\rm MS}} = 6.3710$ [15].

This expansion generates the following approximation to the spectral function $\rho(\sigma)$:

$$\rho(\sigma) = \varrho_0(\sigma) + d_1 \varrho_1(\sigma) + d_2 \varrho_2(\sigma)\,, \tag{6}$$

where the coefficients d_1 and d_2 are the same as in the PT series (5) and the expansion functions are determined by the discontinuity of the corresponding power of the perturbative running coupling, $\varrho_n(\sigma) = \mathrm{Im}[a^{n+1}_{\rm pt}(-\sigma - i\epsilon)]$.

By using the spectral function (6), we obtain the d-function in the form of the expansion (not a power series in a)

$$d_{\rm an}(Q^2) = \Delta^{(1)}_{\rm an}(Q^2) + d_1 \Delta^{(2)}_{\rm an}(Q^2) + d_2\, \Delta^{(3)}_{\rm an}(Q^2)\,, \tag{7}$$

where the $\Delta^{(n)}_{\rm an}$ are analytic functions and $\Delta^{(1)}_{\rm an}(Q^2) = a_{\rm an}(Q^2)$ (see [16] for details).

The Euclidean running coupling $a_{\rm an}(Q^2)$ and the running coupling $\tilde{a}_{\rm an}(s)$ defined in the Minkowskian region are expressed through the function $\varrho_0(\sigma)$ [9, 17]. In the leading order

$$\varrho_0^{(1)}(\sigma) = \frac{1}{\beta_0} \frac{\pi}{\ln^2(\sigma/\Lambda^2) + \pi^2}, \quad \beta_0 = (11 - 2n_f/3)/4\,, \tag{8}$$

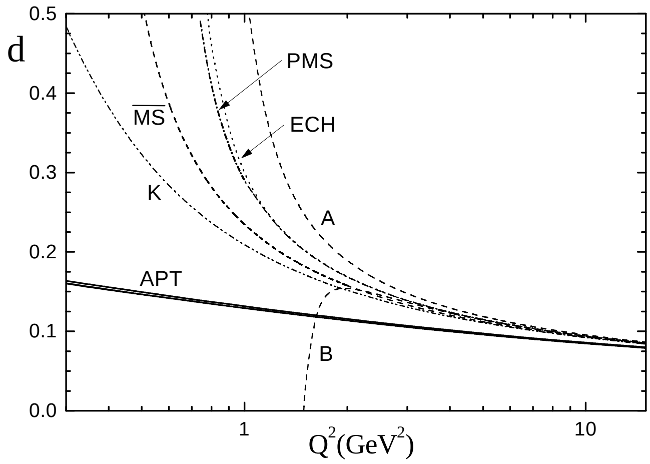

Figure 1. Renormalization scheme dependence of the d-function as a function of Q^2 for the PT and APT approaches. The APT results are shown as solid lines which are very close to each other and practically merge into one curve.

$$a_{\rm an}^{(1)}(Q^2) = \frac{1}{\beta_0}\left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2}\right], \tag{9}$$

$$\tilde{a}_{\rm an}^{(1)}(s) = \frac{1}{\beta_0}\left[\frac{1}{2} - \frac{1}{\pi}\arctan\frac{\ln(s/\Lambda^2)}{\pi}\right]. \tag{10}$$

The expression in the Euclidean region (9) contains the usual logarithmic term that coincides with the perturbation expression containing the ghost pole at $Q^2 = \Lambda^2$. The contribution of this pole is compensated by the second term in Eq. (9) of a power character in Q^2. Written in terms of the initial $a_{\rm pt}$, this term is of the structure of $\exp(-1/a_{\rm pt})$ and therefore makes no contribution to the power series expansion in the coupling $a_{\rm pt}$. That is, the Q^2-power contribution in the Euclidean running coupling (9), invisible in PT, is restored automatically on the basis of the analyticity principle. In contrast to the PT running coupling $a_{\rm pt}(Q^2)$, the analytic function $a_{\rm an}(Q^2)$ has no unphysical singularities: the ghost pole and corresponding branch points (which appear in higher order) are absent. It should be stressed, the APT and PT coincide with each other in the asymptotic region of high energies. A value of the running coupling defined in the Minkowskian region, $\tilde{a}_{\rm an}(s)$, is less than a value of the running coupling in the Euclidean region, $a_{\rm an}(Q^2)$, at the same magnitude of argument [17, 18].

3 Renormalization scheme dependence

A significant source of theoretical uncertainty arises from the RS dependence of the results obtained due to the inevitable inclusion of only a finite number of terms in the PT series. In QCD, that uncertainty is the greater, than smaller a value of typical energy of the process.

There are no general principles that give preference to a particular RS, and in this sense, all schemes are equivalent. The APT method improves this situation and gives very stable results over a wide range of renormalization schemes. To demonstrate this fact, in Fig. 1 we plot functions $d_{\rm pt}(Q^2)$ and $d_{\rm an}(Q^2)$ in different RS. It is seen that predictions in the perturbative approach for $d(Q^2)$ obtained within different RS diverge considerably (see dashed curves A and B). Note should be made of the fact that the schemes A and B are similar to each other and to the optimal PMS [19] and ECH [20] schemes in the sense of the cancellation index [21]: $C_{\rm A} \simeq C_{\rm B} \simeq 2$. For the ECH method, the cancellation index is minimal, equaling unity. The cancellation index for the $\overline{\rm MS}$ scheme turns out to be somewhat bigger, $C_{\overline{\rm MS}} \simeq 3.1$. In Fig. 1, we also draw the curves representing PT results in the PMS, ECH, $\overline{\rm MS}$ and K schemes. For the same schemes, in Fig. 1 we also present results obtained in the APT approach. In this case the scheme arbitrariness is extremely small, and all the curves corresponding to the schemes A, B, PMS, ECH, $\overline{\rm MS}$, and K merge into one thick solid curve. Thus, in the APT, the scheme arbitrariness is very dramatically reduced as compared to that in analogous PT calculations.

4 Ratio R_τ

The total hadronic width of the τ lepton is given by difference of its total width and the partial widths for the electronic and muonic decays $\Gamma(\tau^- \to hadrons\ \nu_\tau) = \Gamma_{tot} - \Gamma_e - \Gamma_\mu$. In an analogy to well-known Drell-ratio for the e^+e^- annihilation into hadrons, one can define a ratio R_τ

$$R_\tau \equiv \frac{\Gamma(\tau^- \to hadrons\ \nu_\tau)}{\Gamma(\tau^- \to \ell\ \bar{\nu}_\ell\ \nu_\tau)}.$$

The theoretical expression for R_τ can be presented as follows

$$R_\tau = 3\left(|V_{ud}|^2 + |V_{us}|^2\right) S_{\rm EW}\,(1+\delta_\tau)\,, \tag{11}$$

where V_{ud} and V_{us} are elements of the CKM quark mixing matrix, $S_{\rm EW}$ is the electroweak factor, and the QCD contribution, δ_τ, is expressed via $r(s)$ as

$$\delta_\tau \,=\, 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+2\frac{s}{M_\tau^2}\right) r(s)\,. \tag{12}$$

This expression is a starting point of our analysis. Within the PT the integral (12) cannot be evaluated directly due to unphysical singularities of the PT running coupling lying in the range of integration.

The most useful trick to rescue the situation is to appeal to analytic properties of the correlator $\Pi(q^2)$. The relations between the functions $r(s)$ and $d(Q^2)$ allow us to represent δ_τ as a contour integral in the complex z plane by choosing the contour to be a circle of radius $|z| = M_\tau^2$ [11]

$$\delta_\tau \,=\, \frac{1}{2\pi i}\oint\limits_{|z|=M_\tau^2} \frac{dz}{z}\left(1-\frac{z}{M_\tau^2}\right)^3\left(1+\frac{z}{M_\tau^2}\right) d(-z)\,. \tag{13}$$

It should be stressed that expressions (12) and (13) are equivalent only when the above-mentioned analytic properties are maintained.

It would seem that the transformation to the contour representation (13) allows one to avoid this difficulty, since in this case unphysical singularities of the running coupling lie outside of the contour, and the procedure of integration can formally be easily accomplished. However, in our opinion, this trick ("sweeping the difficulty under the rug") does by no means solve the problem. Actually, incorrect analytic properties of the running coupling result in Eqs. (12) and (13) for δ_τ being no longer equivalent [14, 22], and, if one remains within PT, nothing can be said about the errors introduced by this transition. The APT may eliminate these problems.

The PT description is based on the contour representation and can be developed in the following two ways. In the Braaten's (Br) method [12] the quantity (13) is represented in the form of truncated power series with the expansion parameter $a_\tau = \bar{\alpha}_s(M_\tau^2)/\pi$. In this case the three-loop representation for δ_τ is

$$\delta_\tau^{\rm Br} = a_\tau \,+ r_1\, a_\tau^2 + r_2\, a_\tau^3\,, \tag{14}$$

where the coefficients r_1 and r_2 in the $\overline{\rm MS}$ scheme with three active flavors are $r_1 = 5.2023$ and $r_2 = 26.366$ [12].

The method proposed by Le Diberder and Pich (LP) [13] uses the PT expansion of the d-function (5). It results to the following non-power representation

$$\delta_\tau^{\rm LP} = A^{(1)}(a) + d_1\, A^{(2)}(a) + d_2\, A^{(3)}(a) \tag{15}$$

with

$$A^{(n)}(a) = \frac{1}{2\pi i}\oint_{|z|=M_\tau^2} \frac{dz}{z}\left(1-\frac{z}{M_\tau^2}\right)^3\left(1+\frac{z}{M_\tau^2}\right) a^n(z)\,. \tag{16}$$

Both these PT approaches, are widely used in the analysis of τ-decay data. However, their status is different. The formula (14) can be obtained self-consistently. In expression (12) one has to use for $r(s)$ the initial perturbative approximation with the expansion parameter a_μ. Then, after integration over s, the logarithmic terms containing $\ln(M_\tau^2/\mu^2)$ are removed by setting $\mu^2 = M_\tau^2$. The same result is obtained if the contour representation (13) is used and the d-function is taken in the form the initial perturbative approximation which preserves the required analytic properties. As for the representation (15), it will be consistent with expressions (12) and (13), if $a(z)$ has analytic properties of the Källén–Lehmann type. The use of the standard PT running coupling with unphysical singularities in (16) breaks this consistency.

The APT description can be equivalently phrased either on the basis of the original expression (12), which involves the Minkowskian quantity $r(s)$, or on the contour representation (13), which involves the Euclidean quantity $d(q^2)$. Within the framework of the APT approach, both forms can be rewritten in terms of the spectral function $\rho(\sigma)$ as [14]

$$\delta_\tau = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma}\rho(\sigma) - \frac{1}{\pi}\int_0^{M_\tau^2} \frac{d\sigma}{\sigma}\left(1-\frac{\sigma}{M_\tau^2}\right)^3\left(1+\frac{\sigma}{M_\tau^2}\right)\rho(\sigma)\,. \tag{17}$$

In Fig. 2, we illustrate the dependence of the R_τ-ratio on the running coupling in the PT(Br) and APT approaches, comparing the convergence properties in the one-loop (dotted lines), two-loop (dashed lines), and three-loop (solid lines) approximations. Numbers above the curves specify the order of the approximation. The

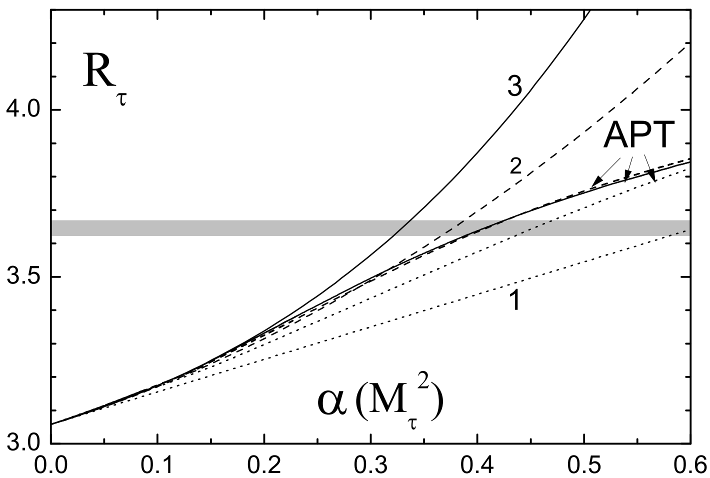

Figure 2. The PT(Br) and APT predictions for the R_τ ratio vs. the running coupling in the $\overline{\mathrm{MS}}$ scheme. The numbers labelling the curves denote the level of the loop expansion used.

shaded area shows the corridor of experimental errors for $R_\tau^{\rm expt} = 3.646 \pm 0.022$ [23]. The convergence properties of the APT expansion seem to be much improved compared to those of the PT expansions.

The APT approach allows one to construct a series for which RS dependence is dramatically reduced. For the hadronic τ decay it is easy for understanding if one takes into account the result which is shown in Fig. 1 for d-functions in different RS: instead of RS unstable and rapidly changing PT results, the APT predictions are practically RS independent.

In the case of massless quarks, the detailed APT analysis of the inclusive τ decay on the three-loop level has been performed in [24]. This investigation together with other results allows us to formulate the following features of the APT method: (i) this approach maintains the correct analytic properties and leads to a self-consistent procedure of analytic continuation from the spacelike to the timelike region; (ii) it has much improved convergence properties and turns out to be stable with respect to higher-loop corrections; (iii) renormalization scheme dependence of the results obtained within this method is reduced dramatically.

5 Vector and axial-vector channels in τ decay

In this section we compare our theoretical result with results that we get from the τ-data presented by the ALEPH Collaboration [3]. These data have been extensively used in various QCD studies including the determination of the strong coupling constant, the test of the conception of quark-hadron duality, the application in the evaluation of the anomalous magnetic moment of the muon and the determination of ρ-meson parameters.

From the complete analysis of the τ branching ratios [3], it is possible to separate the non-strange vector and axial-vector hadronic τ decay channels, $V^-\nu_\tau$ and $A^-\nu_\tau$, respectively. The inclusive observable R_τ-ratio can be written down as

$$R_\tau \,=\, R_{\tau,V} \,+\, R_{\tau,A} \,+\, R_{\tau,S}\,, \tag{18}$$

where $R_{\tau,V}$ and $R_{\tau,A}$ are contributions corresponding to the vector and axial-vector non-strange quark currents, $\Gamma(\tau^- \to {\rm hadrons}_{S=0})$, and $R_{\tau,S}$ includes strange decays, $\Gamma(\tau^- \to {\rm hadrons}_{S=-1})$. Note, for the strange hadronic width vector and axial-vector contributions are not separated so far due to the lack of the corresponding experimental information for the Cabibbo-suppressed modes.

Within the perturbative approximation with massless quarks the vector and axial-vector contributions to R_τ coincide with each other

$$R_{\tau,V} = R_{\tau,A} = \frac{3}{2}|V_{ud}|^2(1\,+\,\delta_\tau)\,, \tag{19}$$

where δ_τ is given by the expression (12). However, the experimental measurements shown that these components are not equal to each other.

Figs. 3 and 4 show the ALEPH measurements for the vector and the axial-vector non-strange quark currents, functions $R_V(s)$ and $R_A(s)$, respectively. These figures clearly demonstrate that the vector current function $R_V(s)$ indicates a dominant large $\rho^-(770)$ resonance and the axial-vector function $R_A(s)$ indicates $a_1(1260)$ resonance. Note that the normalization of the ALEPH functions $v_1(s)$ and $a_1(s)$ differ from that we use here: at the parton-level our function $R(s)$ is equal to 1, therefore $R_{V/A}(s) = 2\,v_1(s)/a_1(s)$.

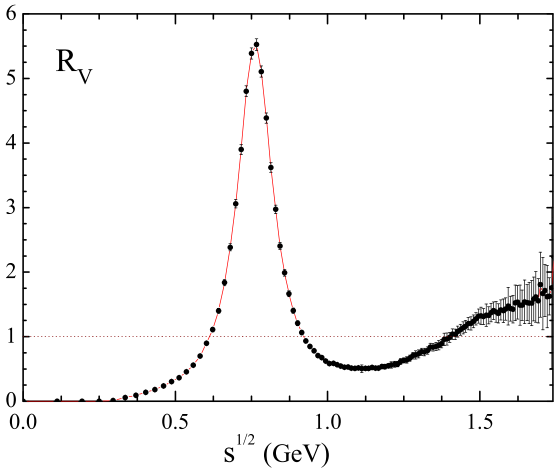

Figure 3. The total inclusive vector current function.

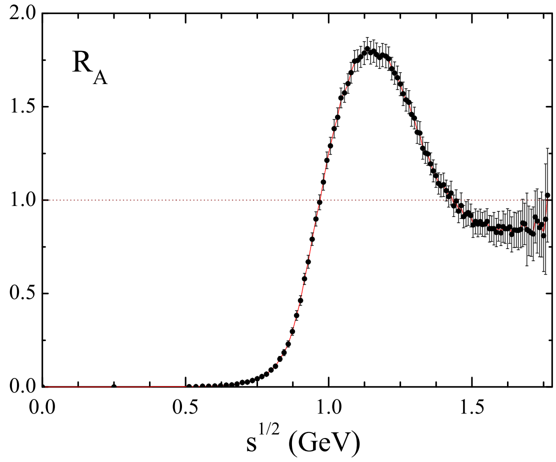

Figure 4. The inclusive τ axial-vector function (without the pion pole).

The ratios $R_{\tau,V}$ and $R_{\tau,A}$ can be obtained through vector and axial-vector functions, $R_V(s)$ and $R_A(s)$, as

$$R^{\rm exp/theor}_{\tau,V/A} = R_0 \int\limits_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+\frac{2s}{M_\tau^2}\right) R^{\rm exp/theor}_{V/A}(s), \tag{20}$$

where $R_0 \equiv 3|V_{ud}|^2 S_{\rm EW}$, $|V_{ud}| = 0.9752 \pm 0.0007$ and $S_{\rm EW} = 1.0194 \pm 0.0040$ (see [3] for details). The experimental value obtained by the ALEPH collaboration for the vector channel is

$$R^{\rm exp}_{\tau,V} = 1.787 \pm 0.013. \tag{21}$$

Based on the APT, involving a summation of threshold singularities [25] and taking into account the nonperturbative character of the light quark masses, as it in details has been described in [26], we take as an input, the value of the running coupling defined in the Minkowskian region in $\overline{\rm MS}$ renormalization scheme $\tilde{a}_s(M_\tau^2) = 0.33$ and reproduce the central experimental value of the ALEPH data for the vector ratio

$$R^{\rm theor}_{\tau,V} = 1.79 = R^{\rm exp\,,centr}_{\tau,V}. \tag{22}$$

The experimental value obtained by the ALEPH collaboration for the total axial-vector channel is $R^{\rm exp,tot}_{\tau,A} = 1.695 \pm 0.013$. The inclusive axial-vector function which show in Fig. 4 does not contain the pion pole. The branching fraction for the $\pi^-\nu$ mode is given as $(10.83 \pm 0.11)\%$ [3]. After subtraction of this pole contribution, we get

$$R^{\rm exp}_{\tau,A1} = 1.087 \pm 0.015. \tag{23}$$

The ALEPH measurements allows us to study the D-function in the non-strange vector and axial-vector channels:

$$D^{\rm exp/theor}_{V/A}(Q^2) = Q^2 \int_0^\infty ds\, \frac{R^{\rm exp/theor}_{V/A}(s)}{(s+Q^2)^2}\,. \tag{24}$$

Although the function $R_{V/A}(s)$ experimentally is not known for all values of s, it is possible to use the following expression

$$R_{V/A}(s) = R^{\rm exp}_{V/A}(s)\,\theta(s_0 - s) + R^{\rm theor}_{V/A}(s)\,\theta(s - s_0)\,, \tag{25}$$

where, for example, the continuum threshold s_0 can be found from the global duality relation [27], that usually gives $s_0 = 1.35 \div 1.75$ GeV^2.

Within the analytic approach the D_V-function has been analyzed in [16]. The improved studying has been done in [26]. These results are close to each other. In complete analogy to vector case [26], we consider the non-strange axial-vector D_A-function. As a first step, we use a simple model for the function $R_A(s)$, that usually used in the QCD sum rules

$$R^{\rm had}_A(s) = \frac{2\pi}{g_A^2}\, m_A^2\, \delta(s - m_A^2) + \left(1 + \frac{\alpha_s^{(0)}}{\pi}\right)\theta(s - s_0). \tag{26}$$

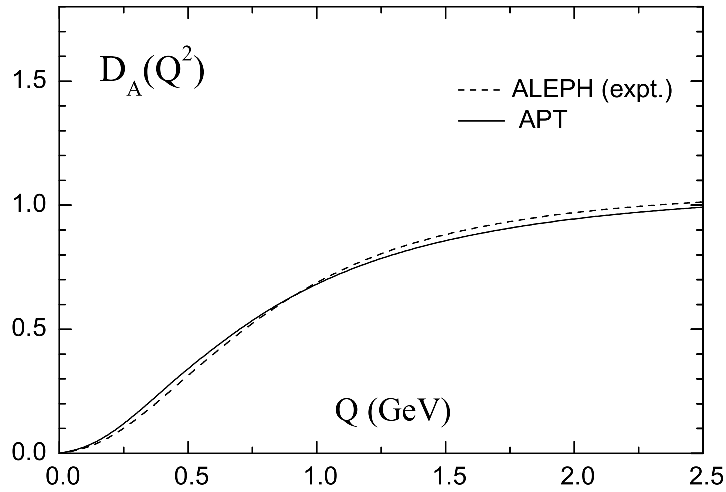

Figure 5. Experimental and theoretical D_A-functions for axial-vector channel.

This model expression leads to the function

$$D_A^{\rm had}(Q^2) = \frac{2\pi}{g_A^2}\,\frac{Q^2\,m_A^2}{(Q^2+m_A^2)^2} + \left(1+\frac{\alpha_s^{(0)}}{\pi}\right)\frac{Q^2}{Q^2+s_0}\,, \tag{27}$$

which reproduces well the "experimental" curve $D_A^{\rm exp}(Q^2)$ constructed by using experimental data shown in Fig. 4 with the parameters: $m_A = 1260$ MeV, $g_A^{-2} = 1.65$, $\alpha_s^{(0)} = 0.4$, and $s_0 = 1.75$ GeV^2. The values of these parameters are close to the parameters that one usually uses in the sum rules method [28].

In Fig. 5 we plot the D_A-function obtained in the APT approach (solid curve) and the experimental curve (dashed line) constructed by using the ALEPH data. The D_A-function turns out to be a smooth function without any traces of resonance structure and, therefore, is useful to use in a theoretical analysis as the Euclidian characteristic of the inclusive process. Fig. 5 demonstrates a good agreement of our result with the experimental curve for whole interval of Q^2. Note here that an use of any finite order of the operator product expansion cannot adequately describe the D-function in the infrared region of low energy scale. The curve corresponding to model expression (27) coincides with the dashed line.

It is important to note, that we obtained the value of $R_{\tau,V}^{\rm theor}$ which agrees well with the experimental data (21) and, at the same time, within the the same theoretical framework, we obtained the value of $R_{\tau,A}^{\rm theor} = 1.045$ which is very close to the experimental value (23).

6 Conclusions

The analytic approach proposed by Shirkov and Solovtsov modifies the perturbative expansions such that the new approximations reflect basic principles of the theory, such as renormalization invariance, spectrality, and causality. Analytic perturbation theory, which was used here, gives a self-consistent description of both the spacelike and timelike regions. This method was applied to describe some physical quantities and functions an experimental information for which can be extracted from the τ lepton decay data.

We performed a comparative analysis of the advantages and disadvantages of different forms of perturbative expansion both from the general standpoint and in the context of application to the inclusive τ decay. We presented the arguments in favor of the APT, which not only agrees with the general principles of the theory but also has a number of practical advantages. In the analytic approach, the two methods for describing the inclusive τ lepton decay in terms of timelike or spacelike variables are equivalent.

Within the APT, the dependence of the results on the choice of the renormalization prescription is essentially reduced, and we can speak of the practical independence of the three-loop expressions from the renormalization scheme. The calculations based on the APT thus considerably reduce the theoretical uncertainty of the results. Therefore, using it as the perturbative component increases the reliability of information about the QCD parameters obtained from the experimental data known with high accuracy for the τ lepton decay.

We considered the Adler function corresponding to the non-strange vector and axial-vector channels. These functions, defined in the Euclidean region, are smooth functions and represent a convenient testing ground for theoretical methods. The conventional method of approximating these function as a sum of perturbative terms and power corrections cannot describe the low energy scale region because both the logarithmic and power expansions diverge at small momenta. We have shown that our approach allows us to describe well the experimental data for τ decay in terms of the $D_{V/A}$-functions down to the lowest energy scale and for R_τ in the non-strange vector axial-vector channels.

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