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MATHEMATICAL ANALYSIS

МАТЕМАТИЧЕСКИЙ АНАЛИЗ

УЧЕБНО-МЕТОДИЧЕСКОЕ ПОСОБИЕ

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The training manual on mathematical analysis contains basic theoretical information on the main sections of the course. The theoretical material is illustrated with examples that allow one to learn the material more deeply. A large number of tasks, contributing to the development of solution skills and to consolidation the acquired theoretical knowledge, are given at the end of each section.

This textbook is intended for students of technical specialties studying in English.

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SECTION 1. LIMITS

1.1. Definition of Numerical Sequence

A *numerical sequence* is defined if there is a rule according to which to every positive integer n corresponds a real number a_n.

In other words, a_n is a function of natural argument n.

The elements of the sequence are called the *terms*. The term a_n is called n-th term (or *general term*) of a sequence. The general terms put into braces denotes a sequence: $\{a_n\}$, $\{b_n\}$, $\{c_n\}$.

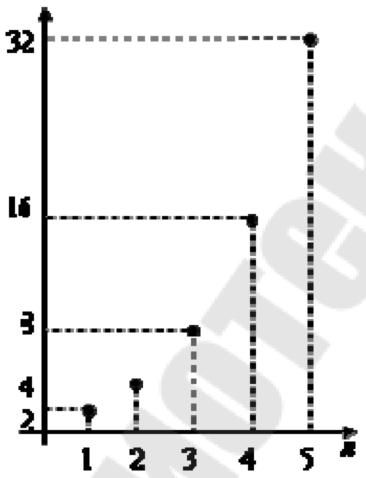
Example 1.1. The general term $a_n = 2^n$ determines an infinite geometric progression with the common ratio 2:

$$\{a_n\} = 2, 4, 8, 16, 32, \ldots .$$

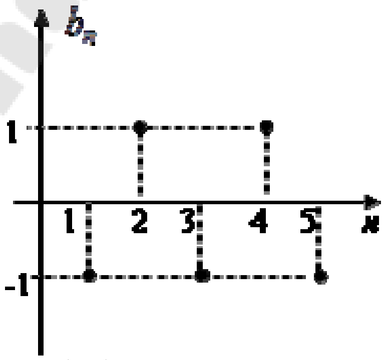
The sequence $\{b_n\} = -1, 1, -1, 1, -1, \ldots$ can be represented by the general term $b_n = (-1)^n$.

The general term of the sequence $\{c_n\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is $c_n = \frac{1}{n}$.

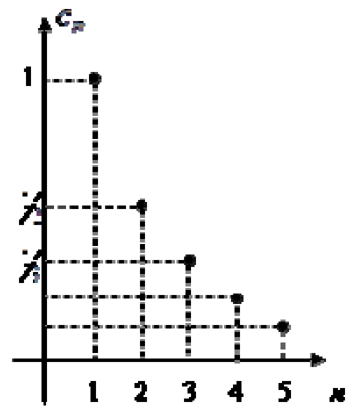
Graphically, the sequence can be represented by points on the number line or as a two domain chart. For example, the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ from the example 1.1 can be graphically represented as follows:



$\{a_n\} = 2, 4, 8, 16, 32, \ldots$



$\{b_n\} = -1, 1, -1, 1, \ldots$



$\{c_n\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

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Bounded Sequences

The sequence $\{a_n\}$ is said to be an *upper bounded sequence* if there exists a finite number of U such that $a_n \le U$ for all natural numbers *n*. The number U is said to be the *upper bound* of $\{a_n\}$.

The sequence $\{a_n\}$ is said to be a *lower bounded sequence* if there exists a finite number of L such that $a_n \ge L$ for all natural numbers *n*. The number L is said to be the *lower bound* of $\{a_n\}$.

A sequence is called *bounded* if there are two finite numbers, L and U, such that $L \le a_n \le U$ for all members of the sequence. Otherwise the sequence is called *unbounded*.

The sequences $\left\{\frac{1}{n}\right\}$ and $\{(-1)^n\}$ are bounded because

$$0 \le \frac{1}{n} \le 1;$$

$$-1 \le (-1)^n \le 1.$$

The sequences $\{2^n\}$ is only lower bounded because $2 \le 2^n$.

The example of unbounded sequence is

$$\{(-2)^n\} = -2;\ 4;\ -8;\ 16;\ -32;\ \ldots\ .$$

Monotone Sequences

A sequence $\{a_n\}$ is called a *monotone increasing* sequence, if $a_{n+1} \ge a_n$ for each natural number *n*. A sequence $\{a_n\}$ is called a *monotone decreasing* sequence, if $a_{n+1} \le a_n$ for each natural number *n*.

The sequence $\left\{\frac{1}{n}\right\}$ is monotone decreasing because

$$a_n = \frac{1}{n}, \quad a_{n+1} = \frac{1}{n+1};$$

$$\frac{1}{n+1} \le \frac{1}{n}.$$

The sequence $\{2^n\}$ is monotone increasing, as

$$a_{n+1} = 2^{n+1} \geq 2^n = a_n.$$

The sequence $\{(-1)^n\}$ is non monotonic.

Exercises

1. Write down a few terms of the following sequences given by general terms:

a) $a_n = \dfrac{n+2}{n+4}$; b) $c_n = \dfrac{n+1}{3^n}$; c) $y_n = \dfrac{2^n}{n!}$;

d) $b_n = \dfrac{(-1)^{n+1}}{2n+1}$; e) $x_n = (-1)^n \dfrac{n}{2n^2-1}$; f) $z_n = \dfrac{(-1)^{n+1}}{(2n-1)!}$.

Which of the given sequences are bounded? Only lower bounded? Only upper bounded?

2. Find the general terms of the following sequences:

a) $\{p_n\} = 1, 3, 5, 7, 9, \ldots$; d) $\{s_n\} = -\dfrac{1}{3}, \dfrac{2}{8}, -\dfrac{3}{13}, \dfrac{4}{18}, \ldots$;

b) $\{q_n\} = \dfrac{1}{3}, \dfrac{2}{9}, \dfrac{3}{27}, \dfrac{4}{81}, \ldots$; e) $\{t_n\} = 1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, \ldots$;

c) $\{r_n\} = 1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \dfrac{1}{25}, \ldots$; f) $\{u_n\} = 2, 0, 6, 0, 10, 0, 14, \ldots$.

Which of the given sequences are bounded? Only lower bounded? Only upper bounded?

1.2. Limit of Numerical Sequence

Number a is called the limit of a sequence $\{a_n\}$ if we can make a_n as close to a as we want for all sufficiently large n.

In mathematical terms "a_n close to a for all sufficiently large n" means that the difference between a_n and a is getting very small, less than arbitrary positive number ε starting from some number N. So we can give a *formal mathematical definition*:

Definition. Number a is called the ***limit*** of a sequence $\{a_n\}$ if for every number $\varepsilon > 0$ there is an integer N such that $|a_n - a| < \varepsilon$ whenever $n > N$.

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The fact that the value a is a limit of a sequence $\{a_n\}$ is symbolically denoted as

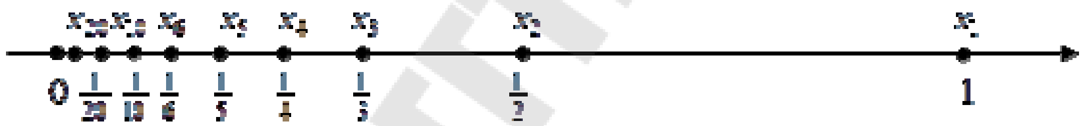
$$\lim_{n\to\infty} a_n = a.$$

Other notations that can be used are $a_n \to a$ as $n \to \infty$ or $\{a_n\} \to a$ as $n \to \infty$.

If the value a is a finite number then the sequence $\{a_n\}$ is called *convergent* ($\{a_n\}$ converges to number a). Otherwise, if a is infinite or doesn't exist, the sequence $\{a_n\}$ is called *divergent*. The sequence $\{a_n\}$ is called *infinitesimal* if $\lim\limits_{n\to\infty} a_n = 0$. The sequence $\{M_n\}$ is called *infinitely large* if $\lim\limits_{n\to\infty} M_n = \infty$.

Example 1.2. Consider the sequence $\left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \cdots$.

Its terms become closer and closer to 0 (see figure), so $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. There is convergent infinitesimal sequence:



Remark. Geometrically, the inequality $|a_n - a| < \varepsilon$ is equivalent to the open interval $(a-\varepsilon; a+\varepsilon)$, which is called the ε-*neighborhood* of a. So we can formulate another definition of the limit:

Number a is called the ***limit*** of a sequence $\{a_n\}$ if every ε-neighborhood of a contains all but a finite number of the terms of $\{a_n\}$.

Example 1.3. The terms of a sequence $\{2^n\} = 2, 4, 8, 16, 32, \ldots, 2^n, \ldots$ get larger and larger with n increasing (see figure), so $\lim\limits_{n\to\infty} 2^n = \infty$. One can say that $\{2^n\}$ is divergent infinitely large sequence.

Example 1.4. The sequence $\{(-1)^n\} = -1, 1, -1, 1, -1, \ldots$ has no limit (see figure), so it is divergent.

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Properties of Sequence limits

Let $\{a_n\}$ and $\{b_n\}$ are both convergent sequences, C is a constant. Then

- The limit of a constant is a constant: $\lim\limits_{n\to\infty} C = C$.
- The constant can be taken out of the limit sign: $\lim\limits_{n\to\infty}(Ca_n) = C\lim\limits_{n\to\infty} a_n$.
- $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$.
- $\lim\limits_{n\to\infty}(a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$.
- $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, if $\lim\limits_{n\to\infty} b_n \neq 0$.

Example 1.5. Evaluate the limit: $\lim\limits_{n\to\infty} \dfrac{n^3+3n-5}{n^3}$.

Solution

$$\lim_{n\to\infty} \frac{n^3+3n-5}{n^3} = \lim_{n\to\infty}\left(\frac{n^3}{n^3}+\frac{3n}{n^3}-\frac{5}{n^3}\right) = \lim_{n\to\infty}\left(1+\frac{3}{n^2}-\frac{5}{n^3}\right) = 1+0-0 = 1.$$

Definition. Two sequences $\{a_n\}$ and $\{b_n\}$ are called *the equivalent* ($a_n \sim b_n$) if the ratio of there terms tends to 1:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

For example, $n^3+3n-5 \sim n^3$ (see the example 1.5).

Theorem 1.1 . If $a_n \sim \alpha_n$,, $b_n \sim \beta_n$ then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\alpha_n}{\beta_n}$.

Theorem 1.1 can be used to calculate the *uncertainties of the form* $\left(\dfrac{\infty}{\infty}\right)$.

Example 1.6. Evaluate the limit $\lim\limits_{n\to\infty} \dfrac{n+5}{\sqrt[3]{8n^3+10}}$.

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Solution

If we "plug" infinity into numerator and denominator we get an indeterminate of the form $\left(\dfrac{\infty}{\infty}\right)$. Replace the expressions in the numerator and denominator with their equivalents:

$$n+5 \sim n,\ \sqrt[3]{8n^3+10} \sim \sqrt[3]{8n^3} = 2n.$$

By the theorem 1.1 we have:

$$\lim_{n\to\infty}\frac{n+5}{\sqrt[3]{8n^3+10}} = \lim_{n\to\infty}\frac{n}{2n} = \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}.$$

Let $P_\alpha(n)$ and $Q_\beta(n)$ be the algebraic expressions, where α and β ($\alpha, \beta > 0$) are the highest powers of $P_\alpha(n)$ and $Q_\beta(n)$ correspondently.

One can use the following rule to calculate $\lim\limits_{n\to\infty}\dfrac{P_\alpha(n)}{Q_\beta(n)}$:

✓ the limit is zero if the power of the numerator α is less than the power of the denominator β;

✓ the limit is infinity if the power of the numerator α is greater than the power of the denominator β;

✓ the limit is equal to the ratio of the coefficients at the highest powers if the powers of the numerator α and the denominator β are equal.

Example 1.7. Evaluate each of the following limits:

a) $\lim\limits_{n\to\infty}\dfrac{\sqrt[4]{n+5}+\sqrt[3]{8n^4}+2n}{n\sqrt[5]{10n^6+n^2}-4\sqrt[6]{n^2+1}}$;   b) $\lim\limits_{n\to\infty}\sqrt{n}\left(\sqrt{n+5}-\sqrt{n-2}\right)$.

Solution

a) Use the theorem 1.1:

$$\sqrt[4]{n+5} \sim \sqrt[4]{n} \sim n^{1/4};$$

$$n\sqrt[5]{10n^6+n^2} \sim n\sqrt[5]{10n^6} = n\sqrt[5]{10}n^{6/5} = \sqrt[5]{10}n^{11/5};$$

$$4\sqrt[6]{n^2+1} \sim 4\sqrt[6]{n^2} = 4n^{1/3};$$

$$\lim_{n\to\infty}\frac{\sqrt[4]{n+5}+\sqrt[3]{8n^4}+2n}{n\sqrt[5]{10n^6+n^2}-4\sqrt[6]{n^2+1}} = \lim_{n\to\infty}\frac{n^{1/4}+\sqrt[3]{8}n^{4/3}+2n}{\sqrt[5]{10}n^{11/5}-4n^{1/3}}.$$

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Now let's compare the powers:

$$\alpha = \max\left\{\tfrac{1}{4}, \tfrac{4}{3}, 1\right\} = \tfrac{4}{3} \Rightarrow n^{1/4} + \sqrt[3]{8}n^{4/3} + 2n \sim 2n^{4/3};$$

$$\beta = \max\left\{\tfrac{11}{5}, \tfrac{1}{3}\right\} = \tfrac{11}{5} \Rightarrow \sqrt[5]{10}n^{11/5} - 4n^{1/3} \sim \sqrt[5]{10}n^{11/5}.$$

Since the power of the numerator is less than the power of the denominator, the limit is equal to zero:

$$\lim_{n\to\infty} \frac{\sqrt[4]{n+5} + \sqrt[3]{8n^4} + 2n}{n\sqrt[5]{10n^6 + n^2} - 4\sqrt[6]{n^2+1}} = \left\{\begin{matrix} \alpha = \dfrac{4}{3}, \beta = \dfrac{11}{5}, \\ \alpha < \beta \end{matrix}\right\} = 0.$$

b) Here we have *an indeterminate of the form* $(\infty - \infty)$. Multiply and divide by conjugate:

$$\lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+5} - \sqrt{n-2}\right) = \lim_{n\to\infty} \frac{\sqrt{n}\left(\sqrt{n+5} - \sqrt{n-2}\right)\left(\sqrt{n+5} + \sqrt{n-2}\right)}{\left(\sqrt{n+5} + \sqrt{n-2}\right)} =$$

$$= \left\{(a-b)(a+b) = a^2 - b^2\right\} = \lim_{n\to\infty} \frac{\sqrt{n}\left((n+5) - (n-2)\right)}{\sqrt{n+5} + \sqrt{n-2}} =$$

$$= \lim_{n\to\infty} \frac{7\sqrt{n}}{\sqrt{n+5} + \sqrt{n-2}} = \left\{\begin{matrix} \sqrt{n+5} \sim \sqrt{n} \\ \sqrt{n-2} \sim \sqrt{n} \end{matrix}\right\} = \lim_{n\to\infty} \frac{7\sqrt{n}}{\sqrt{n} + \sqrt{n}} = \frac{7}{2} = 3{,}5.$$

Exercises

In the exercises 3–16 evaluate each of the following limits:

3. $\lim\limits_{n\to\infty} \dfrac{(1+4n)n^2 - n^4}{(n+3)n^3 + n^4}$.

4. $\lim\limits_{n\to\infty} \dfrac{2n^5 - (3n^2+5)(n^3-3)}{3n^3 + 4n^4 - 1}$.

5. $\lim\limits_{n\to\infty} \dfrac{(n+1)^2 - (n-2)^2}{n^2+1}$.

6. $\lim\limits_{n\to\infty} \dfrac{\left(2n^2+3\right)^2}{\sqrt{4n^8 + 2n^4 + 1}}$.

7. $\lim\limits_{n\to\infty} \dfrac{(2n+1)^3 - n^3}{\sqrt{6n^6+1}}$.

8. $\lim\limits_{n\to\infty} \dfrac{\sqrt{6n^6 + 2n^2 + 3}}{\sqrt[3]{n^6 + 3n^4 + 1}}$.

9. $\lim\limits_{n\to\infty} \dfrac{\sqrt{4n^6 - n + 5}}{(3n+2)\cdot\sqrt[3]{8n^6+1}}$.

10. $\lim\limits_{n\to\infty} \dfrac{7\sqrt[3]{n^5} - 3\sqrt[5]{n^4}}{2\sqrt[3]{n^3} + 3\sqrt[3]{n^2} + 5}$.

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11. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+1}-3n}{7n+3}$.

12. $\lim\limits_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$.

13. $\lim\limits_{n\to\infty}\sqrt[3]{n}\left(\sqrt{n+5}-\sqrt{n-2}\right)$.

14. $\lim\limits_{n\to\infty}\left(n\sqrt{n}-\sqrt{n^3+3n}\right)$.

15. $\lim\limits_{n\to\infty}\dfrac{2^{n+1}-5\cdot 3^n}{3^{n+2}+5\cdot 2^n}$.

16. $\lim\limits_{n\to\infty}\dfrac{3\cdot 7^{n+1}+5^n}{7^{n+2}+5^{n-1}}$.

1.3. Number *e*

heorem .. The sequence $\left\{\left(1+\dfrac{1}{n}\right)^n\right\}$ is monotone increasing bounded sequence. This sequence has a finite limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e, \tag{1.1}$$

where $e=2{,}71828...$ is so-called Euler's Number (the base of natural logarithm).

Remark. The limit (1.1) is used whenever we are dealing with *uncertainty* $\left(1^\infty\right)$. There is a following corollary from formula (1.1):

$$\lim_{n\to\infty}\left(1+\frac{a}{bn+d}\right)^{cn+p}=e^{\frac{a\cdot c}{b}}. \tag{1.2}$$

Example 1.8. Evaluate the limit $\lim\limits_{n\to\infty}\left(\dfrac{n+5}{n+3}\right)^{3n-1}$.

Solution

One have to convert given expression in order to use formula (1.2). One can add and subtract 3 in the numerator:

$$\lim_{n\to\infty}\left(\frac{n+5}{n+3}\right)^{3n-1}=(1^\infty)=\lim_{n\to\infty}\left(\frac{n+3-3+5}{n+3}\right)^{3n-1}=$$

$$=\lim_{n\to\infty}\left(\frac{(n+3)+2}{n+3}\right)^{3n-1}=\lim_{n\to\infty}\left(1+\frac{2}{n+3}\right)^{3n-1}=e^{\frac{2\cdot 3}{1}}=e^6.$$

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Exercises

In the exercises 17–24 evaluate each of the following limits:

17. $\lim\limits_{n\to\infty}\left(\frac{3n+2}{3n-1}\right)^{2n+1}$.

18. $\lim\limits_{n\to\infty}\left(\frac{2n-5}{2n+2}\right)^{n+5}$.

19. $\lim\limits_{n\to\infty}\left(\frac{4n}{4n+3}\right)^{4-n}$.

20. $\lim\limits_{n\to\infty}\left(\frac{n-8}{n+4}\right)^{3n-7}$.

21. $\lim\limits_{n\to\infty}\left(\frac{3n+1}{2n+3}\right)^{3n-1}$.

22. $\lim\limits_{n\to\infty}\left(\frac{5n-4}{2n+1}\right)^{3-n}$.

23. $\lim\limits_{n\to\infty}\left(\frac{n^2+n+5}{n^2+3n-1}\right)^{3n-1}$.

24. $\lim\limits_{n\to\infty}\left(\frac{n^2+5n-1}{n^2+5n+2}\right)^{n+2}$.

1.4. Definition of Function

A function relates each element of a set X with exactly one element of another set Y. The set X is called the *Domain*. The actual values produced by the function is called the *Range*.

The function can be given in different ways:

1) tabular way;

2) graphical way;

3) analytical way.

Types of analytically defined functions:

• An explicit function, when the function is given by the equation $y = f(x)$ to be resolved relative to y.

• An implicit function, when the function is given by an implicit equation as the relation of one of the variables (value) to other variable (argument): $F(x, y) = 0$.

• Parametric function, when the relationship between x and y is realized by a third variable t, called the parameter:

$$\begin{cases} x = x(t), \\ y = y(t). \end{cases}$$

The same function can be specified analytically by listed above methods.

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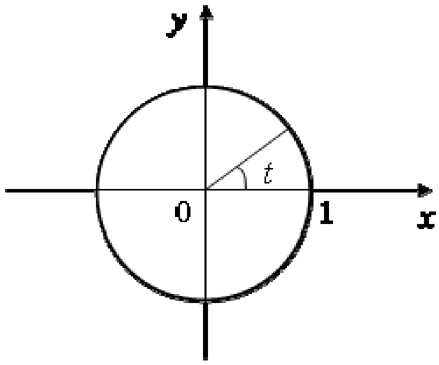
Example 1.9. Consider a unit circle centered at the origin:



An explicit function:

$$y = \pm\sqrt{1 - x^2}.$$

An implicit function:

$$x^2 + y^2 = 1.$$

Parametric function:

$$\begin{cases} x = \cos t, \\ y = \sin t. \end{cases}$$

Parity of Functions

Function is called *even* if $f(-x) = f(x)$ for all x from the domain of the function. Function is called *odd* if $f(-x) = -f(x)$ for all x from the domain of the function.

Geometrically, the graph of an *even* function is symmetric with respect to the *y*-axis. The graph of an *odd* function has rotational symmetry with respect to the origin.

For example function $f(x) = |x| - x^2 + 3$ is even because

$$f(-x) = |-x| - (-x)^2 + 3 = |x| - x^2 + 3 = f(x).$$

Function $f(x) = x - 2x^3 + 3\sin x$ is odd because

$$f(-x) = (-x) - 2(-x)^3 + 3\sin(-x) = -x + 2x^3 - 3\sin x = -f(x).$$

Function $f(x) = x - 3x^2 + 2$ is neither even nor odd because

$$f(-x) = (-x) - 3(-x)^2 + 2 = -x - 3x^2 + 2 \neq f(x);$$

$$f(-x) = -\left(x + 3x^2 - 2\right) \neq -f(x).$$

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Monotonic Functions

For a given function, $y = f(x)$, if the value of y is increasing on increasing the value of x then the function is known as an *increasing function,* and if the value of y is decreasing on increasing the value of x then the function is known as a *decreasing function*.

A *monotonic function* is a function which is either entirely non increasing or non decreasing.

1.5. Limit of Function

We say that the limit of $f(x)$ is L as x approaches a and write this as

$$\lim_{n \to a} f(x) = L$$

provided we can make $f(x)$ as close to L as we want for all x sufficiently close to a from both sides, without actually letting x be a.

For example, consider the limit $\lim\limits_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$. The function $f(x) = \frac{x^2 + 4x - 12}{x^2 - 2x}$ is not defined at the point $x = 2$. Let's calculate the values of the function at points close to $x = 2$:

x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.9985000750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.00015008

So, one can conclude that

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = 4.$$

Properties 1–5 (see subsection 1.2), which are valid for the limits of sequences, also hold for the limits of functions and can be used to calculate limits. By analogy with the concepts of infinitesimal and infinitely large sequences, one can define infinitesimal and infinitely large as x tends to a functions, as well as the concept of equivalent functions.

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Example 1.10. Evaluate each of the following limits:

a) $\lim\limits_{x\to 3}\dfrac{x^2+4x-12}{x^2-2x}$; b) $\lim\limits_{x\to 0}\dfrac{x^2+4x-12}{x^2-2x}$.

Solution

$$\text{a) } \lim_{x\to 3}\frac{x^2+4x-12}{x^2-2x}=\left\{\begin{matrix}\lim\limits_{x\to 3}\left(x^2+4x-12\right)=3^2+4\cdot 3-12=9\\ \lim\limits_{x\to 3}\left(x^2-2x\right)=3^2-2\cdot 3=3\end{matrix}\right\}=\frac{9}{3}=3;$$

$$\text{b) } \lim_{x\to 0}\frac{x^2+4x-12}{x^2-2x}=\left\{\begin{matrix}\lim\limits_{x\to 0}\left(x^2+4x-12\right)=-12\\ \lim\limits_{x\to 0}\left(x^2-2x\right)=0\end{matrix}\right\}=\frac{-12}{0}=\infty.$$

Remark. First we do to calculate $\lim\limits_{x\to a} f(x)$ is plugging $x=a$ into the function. It gives just the answer in some cases (see Example 1.10) but sometimes we get the *indeterminate of the form* $\left(\dfrac{0}{0}\right)$. In this case we have to simplify the function as much as possible by factoring or conjugation. The following formulas are used:

$$a^2-b^2=(a-b)(a+b);$$

$$a^3-b^3=(a-b)(a^2+ab+b^2); \quad a^3+b^3=(a+b)(a^2-ab+b^2);$$

$$(a-b)^2=a^2-2ab+b^2; \quad (a+b)^2=a^2+2ab+b^2.$$

Square trinomial:

$$ax^2+bx+c=a(x-x_1)(x-x_2);$$

$$\text{where } x_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a}, \quad \Delta=b^2-4ac.$$

Example 1.11. Evaluate the limit: $\lim\limits_{x\to 2}\dfrac{x^2+4x-12}{x^2-2x}$.

Solution

Here we have an *indeterminate of the form* $\left(\dfrac{0}{0}\right)$. Let's factorize numerator and denominator:

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$$x^2+4x-12=0;$$

$$\Delta=4^2-4\cdot1\cdot(-12)=16+48=64;$$

$$x_{1,2}=\frac{-4\pm8}{2},\ \ x_1=2,\ \ x_2=-6.$$

After factoring both numerator and denominator we get:

$$\lim_{x\to2}\frac{x^2+4x-12}{x^2-2x}=\lim_{x\to2}\frac{\cancel{(x-2)}(x+6)}{x\cancel{(x-2)}}=\lim_{x\to2}\frac{x+6}{x}=\frac{2+6}{2}=4.$$

Example 1.12. Evaluate the limit: $\lim\limits_{x\to1}\dfrac{3-\sqrt{4x+5}}{2x-2}$.

Solution

$$\lim_{x\to1}\frac{3-\sqrt{4x+5}}{2x-2}=\left(\frac{0}{0}\right).$$

One have to use the conjugation to get rid of the root in the numerator:

$$\lim_{x\to1}\frac{3-\sqrt{4x+5}}{2x-2}=\lim_{x\to1}\frac{\left(3-\sqrt{4x+5}\right)\left(3+\sqrt{4x+5}\right)}{(2x-2)\left(3+\sqrt{4x+5}\right)}=$$

$$=\left\{\begin{matrix}(a-b)(a+b)=a^2-b^2,\\ a=3,\, b=\sqrt{4x+5}\end{matrix}\right\}=\lim_{x\to1}\frac{3^2-\left(\sqrt{4x+5}\right)^2}{2(x-1)\left(3+\sqrt{4x+5}\right)}=$$

$$=\lim_{x\to1}\frac{9-4x-5}{2(x-1)\left(3+\sqrt{4x+5}\right)}=\lim_{x\to1}\frac{4-4x}{2(x-1)\left(3+\sqrt{4x+5}\right)}=$$

$$=\lim_{x\to1}\frac{4\cancel{(1-x)}}{2\cancel{(x-1)}\left(3+\sqrt{4x+5}\right)}=\lim_{x\to1}\frac{-2}{3+\sqrt{4x+5}}=\frac{-2}{3+\sqrt{4+5}}=-\frac{2}{6}=-\frac{1}{3}.$$

Exercises

In the exercises 25–34 evaluate each of the following limits:

25. $\lim\limits_{x\to3}\dfrac{2x^2-3x-9}{x^3-27}$.

26. $\lim\limits_{x\to-3}\dfrac{x^2+6x+9}{x^2+x-6}$.

27. $\lim\limits_{x\to-2}\dfrac{x^3-4x}{4x^2+9x+2}$.

28. $\lim\limits_{x\to4}\dfrac{x^4-16x^2}{x^2-8x+16}$.

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29. $\lim\limits_{x\to 2}\dfrac{x^3+3x^2-10x}{7x^3-10x^2-16}$. 30. $\lim\limits_{x\to -1}\dfrac{x^3+1}{x^3-2x^2+3}$.

31. $\lim\limits_{x\to 2}\dfrac{\sqrt{3x-2}-2}{x^2-4}$. 32. $\lim\limits_{x\to 2}\dfrac{x^3-8}{\sqrt{4x+1}-3}$.

33. $\lim\limits_{x\to -3}\dfrac{\sqrt{x+4}-1}{\sqrt{3-2x}-3}$. 34. $\lim\limits_{x\to 5}\dfrac{\sqrt{1+3x}-\sqrt{2x+6}}{x^2-5x}$.

1.6. Special Limits

1.6.1. Trigonometric Limits

Suppose that the functions $f(x)$ and $g(x)$ are infinitely small as $x\to 0$, the functions $f(x)$ and $g(x)$ contain trigonometric functions.

The indeterminate form

$$`\lim_{x\to 0}\frac{f(x)}{g(x)}=\left(\frac{0}{0}\right)$$

can be calculated by fundamental limit (*first notable limit*):

$$\lim_{x\to 0}\frac{\sin x}{x}=1. \qquad (1.3)$$

The consequences of the formula (1.3):

1) $\lim\limits_{x\to 0}\dfrac{x}{\sin x}=1;$ 4) $\lim\limits_{x\to 0}\dfrac{\operatorname{arctg} x}{x}=1;$

2) $\lim\limits_{x\to 0}\dfrac{\operatorname{tg} x}{x}=1;$ 5) $\lim\limits_{x\to 0}\dfrac{\sin mx}{x}=m;$

3) $\lim\limits_{x\to 0}\dfrac{\arcsin x}{x}=1;$ 6) $\lim\limits_{x\to 0}\dfrac{\sin mx}{\sin nx}=\dfrac{m}{n}.$

These formulas show $\sin x, \operatorname{tg} x, \arcsin x, \operatorname{arctg} x$ to be equivalent x as $x\to 0$, so one can replace this functions by their argument as argument tends to 0.

One can use the following equivalence relations as $x\to 0$:

$$\sin x\sim x; \qquad \arcsin x\sim x;$$

$$\operatorname{tg} x\sim x; \qquad \operatorname{arctg} x\sim x.$$

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Example 1.13. Evaluate each of the following limits:

a) $\lim\limits_{x\to 0}\dfrac{\sin^2 3x+\text{tg}\,2x}{x^2-\text{arctg}\,5x}$;   b) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x\cdot\text{tg}\,3x}$.

Solution

a) $$\lim_{x\to 0}\frac{\sin^2 3x+\text{tg}\,2x}{x^2-\text{arctg}\,5x}=\left(\frac{0}{0}\right)=\left\{\begin{matrix}\sin^2 3x\sim(3x)^2\\ \text{tg}\,2x\sim 2x\\ \text{arctg}\,5x\sim 5x\end{matrix}\right\}=\lim_{x\to 0}\frac{9x^2+2x}{x^2-5x}=$$

$$=\lim_{x\to 0}\frac{9x+2}{x-5}=\frac{9\cdot 0+2}{0-5}=-\frac{2}{5}.$$

b) Use the trigonometric formula:

$$1-\cos x=2\sin^2\frac{x}{2}. \qquad (1.4)$$

Then

$$\lim_{x\to 0}\frac{1-\cos x}{x\cdot\text{tg}\,3x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x\cdot\text{tg}\,3x}=\left\{\begin{matrix}\sin^2\frac{x}{2}\sim\left(\frac{x}{2}\right)^2\\ \text{tg}\,3x\sim 3x\end{matrix}\right\}=\lim_{x\to 0}\frac{2\left(\frac{x}{2}\right)^2}{x\cdot 3x}=$$

$$=\lim_{x\to 0}\frac{\frac{x^2}{2}}{3x^2}=\frac{1}{6}.$$

Remark. According to the formula (1.4) one can say that $1-\cos x\sim\dfrac{x^2}{2}$ as $x\to 0$.

Exercises

In the exercises 35–44 evaluate each of the following limits:

35. $\lim\limits_{x\to 0}\dfrac{\sin^2 2x}{\text{tg}^2 3x}$.

36. $\lim\limits_{x\to 0}\dfrac{\arcsin^3 3x}{x(1-\cos 2x)}$.

37. $\lim\limits_{x\to 0}\sin 3x\cdot\text{ctg}5x$.

38. $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{\sin^2 2x}$.

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39. $\lim\limits_{x\to 0}\dfrac{1-\cos^2 3x}{x\cdot \operatorname{tg} 2x}$.

40. $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{\operatorname{tg} x}\right)$

41. $\lim\limits_{x\to 5}\dfrac{x-5}{\sin(x-5)}$.

42. $\lim\limits_{x\to 1}(1-x)\operatorname{tg}\dfrac{\pi x}{2}$.

43. $\lim\limits_{x\to \pi/2}\dfrac{1-\sin x}{\left(\dfrac{\pi}{2}-x\right)^2}$.

44. $\lim\limits_{x\to 1}\dfrac{\sqrt{x^2-x+1}-1}{\operatorname{tg}\pi x}$.

1.6.2. Limits connected with exponents and logarithms

Uncertainty $\left(1^\infty\right)$ can be revealed with the help of the *second notable limit*:

$$\lim_{\alpha\to 0}(1+\alpha)^{1/\alpha}=e,$$

or

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e.$$

For example, $\lim\limits_{x\to\pi}(1+\sin x)^{1/\sin x}=\left\{\lim\limits_{x\to\pi}\sin x=0\right\}=e.$

The consequences of the second notable limit:

1) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{kx}=e^k;$

2) $\lim\limits_{x\to\infty}\left(1+\dfrac{m}{x}\right)^{x}=e^m;$

3) $\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x}=\dfrac{1}{\ln a};$

4) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a;$

5) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1;$

6) $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1.$

By these formulas the following equivalence relations as $x\to 0$ are hold:

$$\log_a(1+x)\sim\frac{x}{\ln a};\qquad a^x-1\sim x\ln a;$$

$$\ln(1+x)\sim x;\qquad e^x-1\sim x.$$

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Example 1.14. Evaluate each of the following limits:

a) $\lim\limits_{x\to 0}\dfrac{\ln(1+3x)}{5x}$; b) $\lim\limits_{x\to 0}\dfrac{3^{\sin^2 x}-1}{\ln\left(1-\arcsin^2 3x\right)}$.

Solution

a) $\lim\limits_{x\to 0}\dfrac{\ln(1+3x)}{5x}=\left(\dfrac{0}{0}\right)=\{\ln(1+3x)\sim 3x\}=\lim\limits_{x\to 0}\dfrac{3x}{5x}=\dfrac{3}{5}$;

b) $\lim\limits_{x\to 0}\dfrac{3^{\sin^2 x}-1}{\ln\left(1-\arcsin^2 3x\right)}=\left(\dfrac{0}{0}\right)=\left\{\begin{matrix}\sin^2 x\sim x^2\\ \arcsin^2 3x\sim (3x)^2\end{matrix}\right\}=$

$$=\lim_{x\to 0}\frac{3^{x^2}-1}{\ln\left(1-9x^2\right)}==\left\{\begin{matrix}3^{x^2}-1\sim x^2\ln 3\\ \ln\left(1-9x^2\right)\sim -9x^2\end{matrix}\right\}=\lim_{x\to 0}\frac{x^2\ln 3}{-9x^2}=-\frac{1}{9}\ln 3.$$

Exercises

In the exercises 45–54 evaluate each of the following limits.

45. $\lim\limits_{x\to 0}\dfrac{\ln\left(\sin^2 2x+1\right)}{\ln^2(1-2x)}$.

46. $\lim\limits_{x\to 0}\dfrac{e^{3\sin 2x}-1}{1-\cos\sqrt{x}}$.

47. $\lim\limits_{x\to 0}\dfrac{2^{3x}-3^{2x}}{5x}$.

48. $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 3x}{e^{\mathrm{tg}^2 2x}-1}$.

49. $\lim\limits_{x\to 0}\dfrac{\ln(\cos x)}{x\cdot \mathrm{tg}\, 2x}$.

50. $\lim\limits_{x\to 0}\dfrac{\arcsin 2x}{\ln(e-x)-1}$.

51. $\lim\limits_{x\to 1}\dfrac{x^2-1}{\ln x}$.

52. $\lim\limits_{x\to 2}\dfrac{\ln\left(9-2x^2\right)}{\sin 2\pi x}$.

53. $\lim\limits_{x\to \pi}\dfrac{\left(3^{\sin x}-1\right)^2}{\ln\left(2+\cos x\right)}$.

54. $\lim\limits_{x\to 1}\dfrac{2^x-2}{\sin(x^2-1)}$.

1.6.3. Limit of $f(x)^{g(x)}$

The following rules should be followed in calculating the limit $\lim\limits_{x\to a} f(x)^{g(x)}$:

• If $\lim\limits_{x\to a} f(x)=A$, $\lim\limits_{x\to a} g(x)=B$, where A and B are finite numbers, then $\lim\limits_{x\to a} f(x)^{g(x)}=A^B$;

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• If $\lim\limits_{x\to a} f(x) = A > 1$, $\lim\limits_{x\to a} g(x) = +\infty$ then $\lim\limits_{x\to a} f(x)^{g(x)} = \infty$;

• If $\lim\limits_{x\to a} f(x) = A < 1$, $\lim\limits_{x\to a} g(x) = +\infty$ then $\lim\limits_{x\to a} f(x)^{g(x)} = 0$;

• If $\lim\limits_{x\to a} f(x) = 1$, $\lim\limits_{x\to a} g(x) = \infty$ then one can use formula:

$$\lim_{x\to a} f(x)^{g(x)} = \left(1^{\infty}\right) = e^{\lim\limits_{x\to a}(f(x)-1)g(x)}. \qquad (1.5)$$

Example 1.15. Evaluate the limit $\lim\limits_{x\to -1}\left(\dfrac{x+2}{2x+3}\right)^{\frac{1}{x+1}}$.

Solution

By the formula (1.5)

$$\lim_{x\to -1}\left(\frac{x+2}{2x+3}\right)^{\frac{1}{x+1}} = \left\{\begin{array}{l}\lim\limits_{x\to -1}\left(\dfrac{x+2}{2x+3}\right) = \dfrac{-1+2}{-2+3} = 1 \\ \lim\limits_{x\to -1}\dfrac{1}{x+1} = \infty\end{array}\right\} = \left(1^{\infty}\right) =$$

$$= e^{\lim\limits_{x\to -1}\left(\frac{x+2}{2x+3}-1\right)\frac{1}{x+1}} = e^{\lim\limits_{x\to -1}\left(\frac{x+2-2x-3}{2x+3}\right)\frac{1}{x+1}} = e^{\lim\limits_{x\to -1}\left(\frac{-x-1}{2x+3}\right)\frac{1}{x+1}} =$$

$$= e^{\lim\limits_{x\to -1}\frac{-(x+1)}{2x+3}\cdot\frac{1}{x+1}} = e^{\lim\limits_{x\to -1}\frac{-1}{2x+3}} = e^{\frac{-1}{-2+3}} = e^{-1}.$$

In the exercises 55–60 evaluate each of the following limits.

55. $\lim\limits_{x\to 8}\left(\dfrac{2x-7}{2x+1}\right)^{1/\sqrt[3]{x}}$.

56. $\lim\limits_{x\to 1}\left(6-5x^2\right)^{\frac{1}{4x-4}}$.

57. $\lim\limits_{x\to 2}\left(3e^{x-2}-2\right)^{\frac{3x+2}{x-2}}$.

58. $\lim\limits_{x\to \pi/8}\left(\mathrm{tg}\,2x\right)^{\sin\left(\frac{\pi}{8}+x\right)}$.

59. $\lim\limits_{x\to 8}\left(\dfrac{2-x}{x}\right)^{\frac{1}{\ln(2-x)}}$.

60. $\lim\limits_{x\to 0}\left(\dfrac{e^{3x}-1}{x}\right)^{\cos^2(\frac{\pi}{4}+x)}$.

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1.7. One-sided Limits

If x tends to a, remaining all the time *less* than a, then one can say that x *tends to a from the left* and write: $x \to a-0$. If x tends to a, remaining all the time *more* than a, then one can say that x *tends to a from the right* and write: $x \to a+0$.

So we can define *left-handed limit* $f(a-0) = \lim\limits_{x \to a-0} f(x)$ and *right-handed limit* $f(a+0) = \lim\limits_{x \to a+0} f(x)$. Left-handed limit and right-handed limit are called *one-sided limits*.

Example 1.16. Compute the one-sided limits of the function $f(x) = e^{\frac{1}{2-x}}$ as $x \to 2$.

Solution

In the case of left-handed limit $x < 2$ then $2 - x > 0$, so

$$f(2-0) = \lim_{x \to 2-0} e^{\frac{1}{2-x}} = e^{\frac{1}{+0}} = e^{+\infty} = \infty.$$

In the case of right-handed limit $x > 2$ then $2 - x < 0$, so

$$f(2+0) = \lim_{x \to 2+0} e^{\frac{1}{2-x}} = e^{\frac{1}{-0}} = e^{-\infty} = \frac{1}{e^{+\infty}} = 0.$$

1.8. Continuity of Functions

Definition. A function $f(x)$ is said to be *continuous* at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Function is said to be *continuous on the interval* $(a;b)$ if it is continuous at each point in the interval.

Let $f(x)$ be defined in some neighborhood of the point $x = a$ except perhaps the point a itself. The a point is called the *break point* of $f(x)$ if the function is either undefined at the a point or is not continuous at that point.

Let $x = a$ is a break point of $f(x)$. Then it is called:

– the *point of discontinuity of the first type* (or *jump discontinuity*) if there are finite one-sided limits $f(a-0)$ and $f(a+0)$. The value

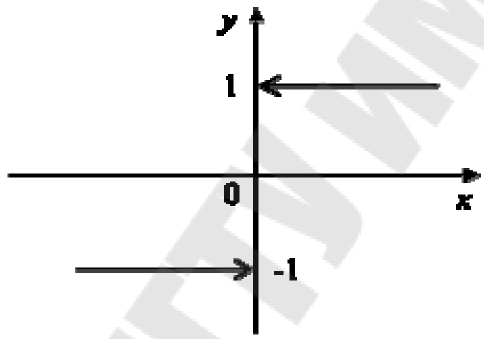
$|f(a-0)-f(a+0)|$ is called the *jump* of the function $f(x)$ at point *a*. If the function jump at a is zero, then a point is called the *removable discontinuity point*;

– the *point of discontinuity of the second type* (or *essential discontinuity*) if at least one of the one-sided limits is infinite or does not exist.

Example 1.17 Function $f(x)=\operatorname{sign} x=\begin{cases}-1 \text{ if } x<0,\\ 1 \text{ if } x>0\end{cases}$ is not defined at the point $a=0$, so $a=0$ is breakpoint. Find one-sided limits as $x\to 0$:

$$\lim_{x\to -0} f(x)=-1;$$
$$\lim_{x\to +0} f(x)=-1,$$

so the point $x_0=0$ is a jump discontinuity point:



Jump of the function:

$$|f(a-0)-f(a+0)|=|-1-1|=2.$$

Example 1.18. Function $f(x)=\dfrac{x^2-9}{x-3}$ is not defined at the point $a=3$. Find one-sided limits as $x\to 3$:

$$\lim_{x\to 3-0} = \frac{x^2-9}{x-3} = \lim_{x\to 3-0} = \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3-0} = (x+3)=6;$$

$$\lim_{x\to 3+0} = \frac{x^2-9}{x-3} = \lim_{x\to 3+0} = \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3+0} = (x+3)=6.$$

A function jump:

$$|f(a-0)-f(a+0)|=0,$$

so $a=3$ is a removable discontinuity point (see Fig. 1.1).

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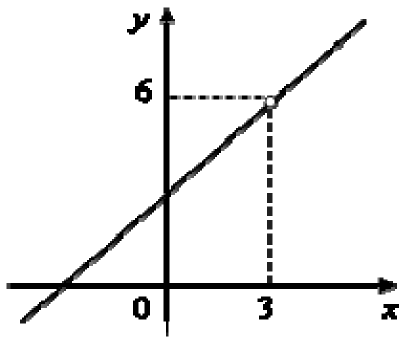

Fig. 1.1

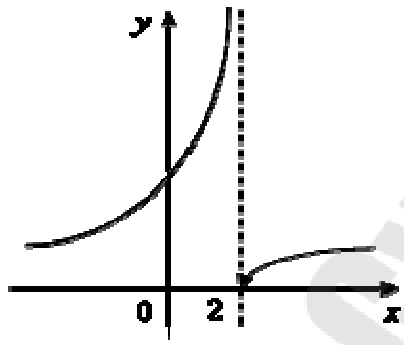

Fig. 1.2

Eample 1.1. Point $a = 2$ is essential discontinuity point for the function $f(x) = e^{\frac{1}{2-x}}$ because $f(2-0) = \infty$ (see example 1.16). The graph of this function is shown in Fig. 1.2.

Exercises

In the exercises 61–66 determine the type of breakpoints of the following functions:

61. $y = \dfrac{\sin(x-2)}{x-2}$.

62. $y = \dfrac{2}{(x-2)^2}$.

63. $y = \dfrac{|x|}{x}$.

64. $y = \dfrac{x^3+1}{x+1}$.

65. $y = \begin{cases} x^2+1, \text{ if } x < 1, \\ 2x, \text{ if } 1 \le x \le 3, \\ x+2, \text{ if } x > 3. \end{cases}$

66. $y = \begin{cases} \dfrac{e^x - 1}{x}, \text{ if } x \le 1, \\ 5^{\frac{1}{x-3}}, \text{ if } x > 1. \end{cases}$

SECTION 2. DIFFERENTIAL CALCULUS

2.1. Definition of the Derivative

Let the function $y = f(x)$ be defined on the segment $[a; b]$. Having fixed a point $x \in (a;b)$, one gives the increment Δx and considers the corresponding *increment of the function* $\Delta y = f(x+\Delta x) - f(x)$.

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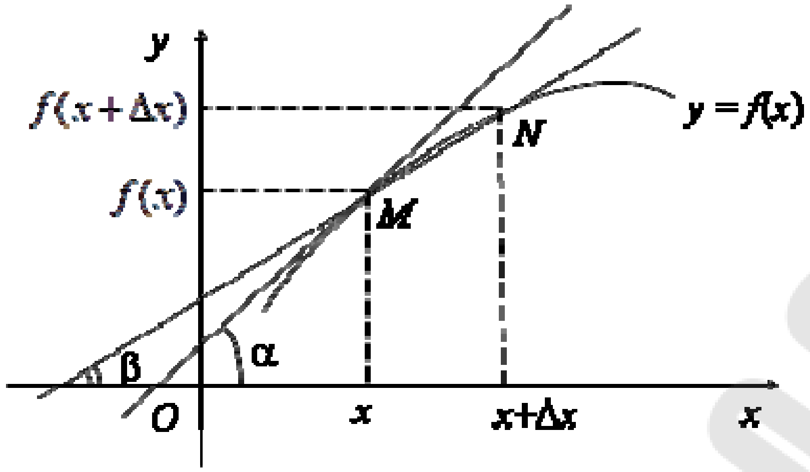

Fig. 2.1

Definition. The *derivative* of $f(x)$ with respect to x is the function $f'(x)$ which is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \qquad (2.1)$$

Alternative notations are $y', y'_x, \frac{df}{dx}, \frac{dy}{dx}$.

If the limit (2.1) exists, then the function $f(x)$ is called *differentiable at the point* x. The function differentiable at each point of the interval $(a; b)$ is called *differentiable in this interval*.

Let's compute the derivative of the function $f(x) = x^2$ using the definition:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \left(2x + \Delta x\right) = 2x.$$

Therefore, $(x^2)' = 2x$.

Figure 2.1 shows that $\operatorname{tg} \beta = \frac{\Delta y}{\Delta x}$. As Δx tends to zero, the secant MN becomes tangent line to the graph of the function $y = f(x)$ at the point $M(x, f(x))$. This implies the *geometric meaning of the derivative*: the derivative at the point x is equal to the tangent of the angle α of the slope of the tangent line drawn at the point $M(x, y)$ to the graph of the function $y = f(x)$.

Based on the geometric meaning of the derivative, we can write the tangent line equation at the point x_0:

$$y = y_0 + f'(x_0)(x - x_0).$$

If the function $y = f(x)$ describes any physical process, then the derivative y' characterizes the rate of this process – this is the *physic meaning of the derivative*. In particular, if the position of an object after t units of time is given by $f(t)$ then the instantaneous velocity of the object at $t = a$ is given by $f'(a)$.

If C is a constant, and $u(x)$ and $v(x)$ are some differentiable functions, then the following *differentiation rules* are valid:

1) $C' = 0$ (constant rule);

2) $x' = 1$;

3) $(u \pm v)' = u' \pm v'$ (sum rule);

4) $(Cu)' = Cu'$;

5) $(u \cdot v)' = u' \cdot v + u \cdot v'$ (product rule);

6) $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$ (quotient rule);

7) if $f(x) = f(u(v(x)))$, then $y'_x = f'_u \cdot u'_x$ (chain rule).

Based on the definition of the derivative and the *differentiation* rules it is possible to compile a table of derivatives of the main elementary functions.

Derivative table

1. $(u^\alpha)' = \alpha u^{\alpha-1} \cdot u'$	10. $(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$
2. $(e^u)' = e^u \cdot u'$	11. $(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$
3. $(a^u)' = a^u \ln a \cdot u'$	12. $(\mathrm{arctg}\, u)' = \frac{1}{1+u^2} \cdot u'$
4. $(\ln u)' = \frac{1}{u} \cdot u'$	13. $(\mathrm{arcctg}\, u)' = -\frac{1}{1+u^2} \cdot u'$

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5. $(\log_a u)' = \dfrac{1}{u \ln a} \cdot u'$	14. $\left(\dfrac{1}{u}\right)' = -\dfrac{1}{u^2} \cdot u'$
6. $(\sin u)' = \cos u \cdot u'$	15. $(\sqrt{u})' = \dfrac{1}{2\sqrt{u}} \cdot u'$
7. $(\cos u)' = -\sin u \cdot u'$	16. $(\text{sh } u)' = \text{ch } u \cdot u'$
8. $(\text{tg } u)' = \dfrac{1}{\cos^2 u} \cdot u'$	17. $(\text{ch } u)' = \text{sh } u \cdot u'$
9. $(\text{ctg } u)' = -\dfrac{1}{\sin^2 u} \cdot u'$	18. $(\text{th } u)' = \dfrac{1}{\text{ch}^2 u} \cdot u'$
19. $(\text{cth } u)' = -\dfrac{1}{\text{sh}^2 u} \cdot u'$	

Example 2.1 . Find derivatives of the following functions:

a) $y = 2x^4 + 5x - \dfrac{2}{3} + \dfrac{4}{x^3} + \dfrac{x^2}{3\sqrt{x}}$; b) $y = (x^2+1) \cdot \text{arctg } x$;

c) $y = \sin^3 2x;$; d) $y = \dfrac{\ln(x^2+1)}{x^2}$.

Solution

a) Transform the function using the formulas

$$\frac{1}{x^m} = x^{-m}, \quad \sqrt[n]{x^m} = x^{m/n}.$$

Then $y = 2x^4 + 5x - \dfrac{2}{3} + 4x^{-3} + \dfrac{1}{3}x^{3/2}$.

Now use a table of derivatives, sum rule and constant rule:

$$y' = 8x^3 + 5 - 0 + 4 \cdot (-3)x^{-4} + \frac{1}{3} \cdot \frac{3}{2} x^{1/2} = 8x^3 + 5 - \frac{12}{x^4} + \frac{1}{2}\sqrt{x}.$$

b) Use a table of derivatives and product rule:

$$y' = \left((x^2+1) \cdot \text{arctg } x\right)' = (x^2+1)' \cdot \text{arctg } x + (x^2+1) \cdot (\text{arctg } x)' =$$

$$= 2x \cdot \text{arctg } x + (x^2+1) \cdot \frac{1}{1+x^2} = 2x \cdot \text{arctg } x + 1.$$

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c) Here we have a composition of three function – power function, sine and polynomial function $2x$, – therefore we use chain rule:

$$y' = \left(\sin^3 2x\right)' = \left((\sin 2x)^3\right)' = 3(\sin 2x)^2(\sin 2x)' =$$

$$= 3\sin^2 2x \cdot \cos 2x \cdot (2x)' = 3\sin^2 2x \cos 2x \cdot 2 = 6\sin^2 2x \cos 2x.$$

d) To compute this derivative we use quotient rule:

$$y' = \left(\frac{\ln(x^2+1)}{x^2}\right)' = \frac{\left(\ln(x^2+1)\right)' \cdot x^2 - \ln(x^2+1) \cdot \left(x^2\right)'}{\left(x^2\right)^2} =$$

$$= \frac{\frac{1}{x^2+1} \cdot 2x \cdot x^2 - \ln(x^2+1) \cdot 2x}{x^4} = \frac{\frac{2x^3}{x^2+1} - 2x\ln(x^2+1)}{x^4} =$$

$$= \frac{2}{x(x^2+1)} - \frac{2\ln(x^2+1)}{x^3}.$$

Exercises

67. Find the derivatives of the following functions using the definition:

a) $y = \sqrt{x}$; b) $y = \ln x$; c) $y = \sin x$.

In the exercises 68–111 find the derivatives of given functions:

68. $y = 2x^5 + 3x^2 - 4x + 8.$

69. $y = 3x^5 - \frac{x}{5} + 2x - 1.$

70. $y = 3x^3 - \frac{x^2}{5} + 0{,}5\sqrt{x} - \ln 3.$

71. $y = \left(x^3 + 3\right)\left(4 - \frac{1}{x^2}\right).$

72. $y = ax^3 + bx^2 - cx + d.$

73. $y = \frac{x^3}{3} + \frac{3}{x^3} + \cos 1.$

74. $y = \sqrt[3]{x^5} + \frac{4}{x^7} - \frac{1}{2\sqrt[5]{x^2}} - e^2.$

75. $y = \sqrt[4]{x} + \frac{\sqrt[3]{x^2}}{2} - 2x\sqrt{x}.$

76. $y = 3\sqrt{x} + \frac{5}{\sqrt{x}} - \frac{2}{\sqrt[3]{x}} - \frac{x}{7}.$

77. $y = \frac{x^2 - \sqrt{x} - 1}{2x^3}.$

78. $y = \frac{x^3 - 4\sqrt{x} + x^2 + 2}{x^2}.$

79. $y = \frac{\left(x^2 - 3\right)^2}{x}.$

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80. $y=\dfrac{3x^2-x+5}{2}.$

81. $y=\dfrac{\cos x\cdot\arccos x}{5}.$

82. $y=\sqrt[3]{x}\left(2+x^2-\sqrt{x}\right).$

83. $y=(3x-2)\left(2\sqrt{x}-\sqrt[3]{x}\right).$

84. $y=\dfrac{1+\sqrt{x}}{1-\sqrt{x}}.$

85. $y=\dfrac{2+x}{3-x}.$

86. $y=x^3\cdot 3^x.$

87. $y=x\ln x.$

88. $y=x\cdot\sin x-\cos x.$

89. $y=\dfrac{\arccos x}{\arcsin x}.$

90. $y=(3x+1)^{10}.$

91. $y=2\sin(3x+5).$

92. $y=e^{\sin 2x}+\arcsin 3x.$

93. $y=\operatorname{arctg} e^{x/2}.$

94. $y=\left(x^3+\dfrac{1}{x}-2\right)^4.$

95. $y=\dfrac{1+x}{\sqrt{1-2x}}.$

96. $y=\sqrt{1-x^2}.$

97. $y=\log_2(x^2+2).$

98. $y=\dfrac{1}{(3x+1)^2}+\sqrt[4]{\left(x^2+2x-4\right)^3}.$

99. $y=(2\sqrt{2x}-3)^3.$

100. $y=(x+2)^4\cdot\sin 5x.$

101. $y=\dfrac{1}{\sin(2/x)}.$

102. $y=\sqrt{\ln(\cos 3x)}.$

103. $y=\sqrt[5]{\cos^2 3x}.$

104. $y=\arccos\dfrac{3x+1}{\sqrt{2}}.$

105. $y=\dfrac{5}{\operatorname{tg}^2 2x}.$

106. $y=\sin^3 5x.$

107. $y=\log_6^4(1-x^3).$

108. $y=\dfrac{\operatorname{tg}^3 x}{3}-\operatorname{tg} x+x.$

109. $y=\cos x-\dfrac{\cos^3 x}{3}.$

110. $y=\log_5\log_3(2x+1).$

111. $y=\ln(x+\sqrt{x^2-1}).$

112. Prove that the function $y=\ln\dfrac{1}{1+x}$ satisfies the relation $xy'+1=e^y.$

113. Write the tangent line equations to the hyperbola $y=\dfrac{1}{x}$ at the point $x=3.$

114. Two points move in a straight line according to the laws $s_1=t^3-3t$ and $s_2=t^3-5t^2+17t-4.$ When will their speeds be equal?

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2.2. Parametric Functions

Let x and y are given as functions of the variable t:

$$\begin{cases} x = x(t), \\ y = y(t). \end{cases}$$

Then it is said that the function $y = f(x)$ is set *parametrically* (variable t is called a *parameter*). If $x(t)$ and $y(t)$ are differentiable and $x_t' \neq 0$ then the derivative y_x' can be found by the formula:

$$y_x' = \frac{y_t'(t)}{x_t'(t)}. \qquad (2.2)$$

Example 2.2 . Find the derivative y_x' of the parametrical function

$$\begin{cases} x = \operatorname{arctg} e^t, \\ y = e^{-t}. \end{cases}$$

Solution

a) Find x_t' and y_t':

$$x_t' = \frac{1}{1+\left(e^t\right)^2} \cdot e^t = \frac{e^t}{1+e^{2t}},, \quad y_t' = e^{-t} \cdot (-1) = -e^{-t}.$$

b) By formula (2.2) we get

$$y_x' = e^{-t} : \frac{e^t}{1+e^{2t}} = \frac{e^{-t}(1+e^{2t})}{e^t} = \frac{e^{-t}+e^t}{e^t} = e^{-2t}+1.$$

Exercises

In the exercises 115–118 find the derivatives of parametric functions:

115. $\begin{cases} x = \dfrac{t+1}{t}, \\ y = \dfrac{t-1}{t}. \end{cases}$

116. $\begin{cases} x = \operatorname{arctg} e^{t/2}, \\ y = \sqrt{e^t+1}. \end{cases}$

117. $\begin{cases} x = \ln(1+t^2), \\ y = t - \operatorname{arctg} t. \end{cases}$

118. $\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t. \end{cases}$

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2.3. Higher Order Derivatives

A *derivative of the second order* or *second derivative* of a function $f(x)$ is called a derivative of its derivative $f'(x)$:

$$f''(x) = \left(f'(x)\right)'.$$

Similarly, derivatives of the third, fourth and, generally, of any n-th order are defined:

$$f'''(x) = \left(f''(x)\right)',\ f^{IV}(x) = \left(f'''(x)\right)',\ \ldots,\ f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'.$$

Collectively the second, third, *etc*. derivatives are called ***higher order derivatives***.

Example 2.3. Find the first four derivatives for the function $f(x) = 5x^3 - 2x^2 + 4x - 1$.

Solution

$$f'(x) = \left(5x^3 - 2x^2 + 4x - 1\right)' = 15x^2 - 4x + 4;$$

$$f''(x) = \left(15x^2 - 4x + 4\right)' = 30x - 4;$$

$$f'''(x) = \left(30x - 4\right)' = 30;$$

$$f^{(4)}(x) = \left(30\right)' = 0.$$

Exercises

In the exercises 119–124 find the second derivative for each of the following functions:

119. $y = xe^{x^2}$.

120. $y = \ln(x + \sqrt{1 + x^2})$.

121. $y = \dfrac{1}{1 + 4x^2}$.

122. $y = 2^{\sqrt{x}}$.

123. $y = a^x,\ y^{(n)} - ?$

124. $y = xe^x,\ y^{(n)} - ?$

2.4. Differentials

For any differentiable function $y = f(x)$ the increment

$$\Delta y = f(x + \Delta x) - f(x)$$

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can be represented as

$$\Delta y = f'(x)\Delta x + \alpha(x)$$

where $\alpha(x)$ is infinitesimal as $\Delta x \to 0$.

Definition. The main linear with a respect to Δx part of the function increment is called the ***differential***:

$$dy = f'(x)\Delta x.$$

As the differential of the independent value x is equal its increment ($dx = \Delta x$) one can write

$$dy = f'(x)dx. \tag{2.3}$$

For example, $d\big(\ln(2x+1)\big) = \dfrac{1}{2x+1}\cdot 2\cdot dx = \dfrac{2dx}{2x+1}$.

To calculate the differential, apply the rules similar to the rules for calculating derivatives:

$$d(Cu) = Cdu; \qquad d(u \pm v) = du \pm dv;$$

$$d(uv) = vdu + udv, \qquad d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}.$$

Application of differential to approximate calculations

If we think of Δx as the change in x_0 then $\Delta y = f(x_0 + \Delta x) - f(x_0)$ is the change in y corresponding to the change in x_0. Now, if Δx is small we can assume that $\Delta y \approx dy = f'(x_0)\Delta x$, where $\Delta x = x - x_0$.

So finally we obtain a formula for the approximate calculation of the value of the function at a point x close to the point x_0:

$$f(x) \approx f(x_0) + f'(x_0)\Delta x. \tag{2.4}$$

Example 2.4. Calculate approximately $\sqrt[4]{85}$.

Solution

Let's consider $f(x) = \sqrt[4]{x}$. One knows the precise value $f(81) = \sqrt[4]{81} = 3$. So let's take

$$x_0 = 81, \quad x = 85, \quad \Delta x = 85 - 81 = 4.$$

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As

$$f'(81)=\frac{1}{4}x^{-3/4}\Big|_{x=81}=\frac{1}{4\sqrt[4]{x^3}}\Big|_{x=81}=\frac{1}{4\sqrt[4]{81^3}}=\frac{1}{108},$$

then by formula (2.4) we finally get:

$$\sqrt[4]{85}\approx 3+\frac{1}{108}\cdot 4\approx 3{,}037.$$

Exercises

In the exercises 125–127 find the differential for each of the following functions:

125. $y=\frac{x^2+5}{2}$. 126. $y=2^{-\frac{1}{\cos x}}$. 127. $y=\arccos\frac{1}{\sqrt{x}}$.

In problems 128–130, calculate approximately using the differential:

128. $\sqrt{8{,}76}$. 129. $\sin 35^\circ$. 130. $\ln 1{,}1$.

2.5. L'Hospital's Rule

Let $f(x)$ and $g(x)$ are differentiable functions. If both $f(x)$ and $g(x)$ are infinitesimal or infinitely large as $x\to a$ where a can be any real number, infinity or negative infinity, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

This algorithm is named the *L'Hospital's Rule*.

Example 2.5. Calculate $\lim\limits_{x\to 0}\frac{\ln x}{\operatorname{ctg} x}$.

Solution

As

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}\ln x=-\infty \text{ and } \lim_{x\to 0}g(x)=\lim_{x\to 0}\operatorname{ctg} x=\infty,$$

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then one can use L’Hospital’s Rule:

$$\lim_{x\to 0}\frac{\ln x}{\operatorname{ctg} x}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{(\ln x)'}{(\operatorname{ctg} x)'}=-\lim_{x\to 0}\frac{1/x}{1/\sin^2 x}=-\lim_{x\to 0}\frac{\sin^2 x}{x}=$$

$$=-\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\sin x=\left\{\begin{matrix}\text{the first notable limit}\\ \lim\limits_{x\to 0}\frac{\sin x}{x}=1\end{matrix}\right\}=1\cdot 0=0.$$

The L’Hospital’s Rule can also be applied in the case of intermediate form $(0\cdot\infty)$.

Let’s $\lim\limits_{x\to a} f(x)=0$, $\lim\limits_{x\to a} g(x)=\infty$, then

$$\lim_{x\to a} f(x)\cdot g(x)=(0\cdot\infty)=\lim_{x\to a}\frac{f(x)}{(g(x))^{-1}}=\left(\frac{0}{0}\right),$$

or

$$\lim_{x\to a} f(x)\cdot g(x)=(0\cdot\infty)=\lim_{x\to a}\frac{g(x)}{(f(x))^{-1}}=\left(\frac{\infty}{\infty}\right).$$

Eample 2.6. Calculate $\lim\limits_{x\to 0+0} x\ln x$.

Solution

As $\lim\limits_{x\to 0+0}\ln x=-\infty$, then we have uncertainty $(0\cdot\infty)$. Transform function to use L’Hospital’s Rule:

$$\lim_{x\to 0+0} x\ln x=(0\cdot\infty)=\lim_{x\to 0+0}\frac{\ln x}{1/x}=\lim_{x\to 0+0}\frac{(\ln x)'}{(1/x)'}=\lim_{x\to 0+0}\frac{1/x}{-\frac{1}{x^2}}=$$

$$=-\lim_{x\to 0+0}\frac{x^2}{x}=-\lim_{x\to 0+0} x=0.$$

The indeterminate forms (∞^0), (0^∞), (1^∞) can be reduced to uncertainty $(0\cdot\infty)$ by using logarithm and its properties:

$$\lim_{x\to a}(f(x))^{g(x)}=\lim_{x\to\infty} e^{\ln(f(x))^{g(x)}}=\lim_{x\to\infty} e^{g(x)\ln f(x)}=e^{\lim\limits_{x\to\infty} g(x)\ln f(x)}.$$

Example 2.7. Calculate $\lim\limits_{x\to\pi/2}(\text{tg }x)^{\text{ctg }x}$.

Solution

$$\lim_{x\to\pi/2} f(x) = \lim_{x\to\pi/2} \text{tg }x = \infty.$$

We have indeterminate form (∞^0). Use logarithm properties:

$$\lim_{x\to\pi/2}(\text{tg }x)^{\text{ctg }x} = \lim_{x\to\pi/2} e^{\ln(\text{tg }x)^{\text{ctg }x}} = \lim_{x\to\pi/2} e^{\text{ctg }x\cdot\ln(\text{tg }x)} = e^{\lim\limits_{x\to\pi/2}[\text{ctg }x\cdot\ln(\text{tg }x)]}.$$

Find the limit of exponent:

$$\lim_{x\to\pi/2}[\text{ctg }x\cdot\ln(\text{tg }x)] = (0\cdot\infty) = \lim_{x\to\pi/2}\frac{\ln(\text{tg }x)}{\text{tg }x} = \left(\frac{\infty}{\infty}\right) =$$

$$= \lim_{x\to\pi/2}\frac{(\ln(\text{tg }x))'}{(\text{tg }x)'} = \lim_{x\to\pi/2}\frac{\frac{1}{\text{tg }x}\cdot\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}} = \lim_{x\to\pi/2}\frac{1}{\text{tg }x} = 0.$$

Therefore

$$\lim_{x\to\pi/2}(\text{tg }x)^{\text{ctg }x} = e^{\lim\limits_{x\to\pi/2}[\text{ctg }x\cdot\ln(\text{tg }x)]} = e^0 = 1.$$

Exercises

In the exercises 138–169 evaluate limits using L'Hospital's Rule:

131. $\lim\limits_{x\to 2}\dfrac{x^3-5x^2+8x-4}{x^3-3x^2+4}$.

132. $\lim\limits_{x\to 1}\dfrac{\ln x}{1-x^4}$.

133. $\lim\limits_{x\to 2}\dfrac{\sqrt{x^2+1}-\sqrt{5}}{\sqrt{x^2+5}-3}$.

134. $\lim\limits_{x\to 3\pi}\dfrac{\text{tg }x}{\text{tg }6x}$.

135. $\lim\limits_{x\to 0}\dfrac{1-\cos ax}{1-\cos bx}$.

136. $\lim\limits_{x\to 0}\dfrac{5^x-1}{2x^2+3x}$.

137. $\lim\limits_{x\to 0}\dfrac{\ln(\cos ax)}{\ln(\cos bx)}$.

138. $\lim\limits_{x\to 0}\dfrac{\ln(\sin ax)}{\ln(\sin bx)}$.

139. $\lim\limits_{x\to\infty}\dfrac{x^2-\sin x}{\cos x-x^2}$.

140. $\lim\limits_{x\to 0}\dfrac{\ln(1-3x)}{\text{arctg }2x}$.

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141. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\cos x}{(x-\pi/2)^2}$.

142. $\lim\limits_{x\to 0} \frac{x-\operatorname{arctg} x}{x^2}$.

143. $\lim\limits_{x\to 0} \frac{\operatorname{tg} x - x}{x-\sin x}$.

144. $\lim\limits_{x\to +\infty} \frac{\ln(x+5)}{\sqrt[3]{x+2}}$.

145. $\lim\limits_{x\to +\infty} x^{10}\cdot e^{-x}$.

146. $\lim\limits_{x\to \infty} x\sin\frac{a}{x}$.

147. $\lim\limits_{x\to 0} \operatorname{ctg} x \ln(x+e^x)$.

148. $\lim\limits_{x\to +\infty} (\ln x)^{1/x}$.

149. $\lim\limits_{x\to +\infty} (x+1)^{1/(1+\ln x)}$.

150. $\lim\limits_{x\to 3} (10-3x)^{2/(x^2-9)}$.

151. $\lim\limits_{x\to 1} (1-x)^{\cos(\pi x/2)}$.

152. $\lim\limits_{x\to 1} (1+\sin \pi x)^{\operatorname{ctg} \pi x}$.

153. $\lim\limits_{x\to +0} \sqrt[x]{e^x + x}$.

154. $\lim\limits_{x\to 0} \sqrt[x]{\cos\sqrt{x}}$.

2.6. Study of Functions Using Derivatives

I. Monotonicity intervals. Extrema.

A function is said to be monotonic on an interval $(a;b)$ if it does not increase (decreases) everywhere on the given interval.

Theorem (monotonic sufficient conditions):

1. If a differentiable function $y=f(x)$ has ***positive derivative*** on the interval $(a;\,b)$ then $y=f(x)$ ***increases*** on $(a;\,b)$.

2. If a differentiable function $y=f(x)$ has ***negative derivative*** on the interval $(a;\,b)$ then $y=f(x)$ ***decreases*** on $(a;\,b)$.

The intervals over which the derivative of the function retains a certain sign are called *the intervals of the monotony* of the function.

The function $y=f(x)$ is said to have ***global maximum global minimum*** at a point $x=c$ if $f(x)\le f(c)$ $\left(f(x)\ge f(c)\right)$ for every x *from the Domain*.

The function $y=f(x)$ is said to have ***a local maximum*** (***local minimum*** at a point $x=c$ if $f(x)\le f(c)$ $\left(f(x)\ge f(c)\right)$ for every x *in some open interval around* $x=c$.

We will collectively call the minimum and maximum points the extrema of the function.

A necessary condition for the existence of an extremum. Let the function $y=f(x)$ be defined in some neighborhood of the point $x=c$.

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If the function $y = f(x)$ has an extremum at the point $x = c$, then $f'(c) = 0$ or doesn't exist.

Points satisfying the necessary condition are called *critical points*.

Sufficient conditions for the existence of an extremum at a critical point.

First Derivative Test . If, when passing through a critical point $x = c$, the derivative $f'(x)$ changes sign from «+» to «–» then the function has a local maximum at the point $x = c$. If, when passing through a critical point $x = c$, the derivative $f'(x)$ changes sign from «–» to «+» then the function has a local minimum at the point $x = c$. If $f'(x)$ is the same sign on both sides of $x = c$ then it is neither a local maximum nor a local minimum.

High Order Derivative Test . Suppose that $x = c$ is a critical point of $y = f(x)$ such that $f'(c) = 0$ and higher order derivatives up to the second order are continuous in a neighborhood of the point $x = c$ and

$$f'(c) = f''(c) = f'''(c) = \cdots = f^{(2n-1)}(c) = 0,\ f^{(2n)}(c) \neq 0.$$

Then if $f^{(2n)}(c) > 0$ then $f(x)$ has *local minimum* at $x = c$; if $f^{(2n)}(c) < 0$ then $f(x)$ has *local maximum* at $x = c$.

Example 2.8. Find and classify all the critical points of the function $y = x^2 + \dfrac{4}{x^2}$. Give the intervals where the function is monotonic.

Solution

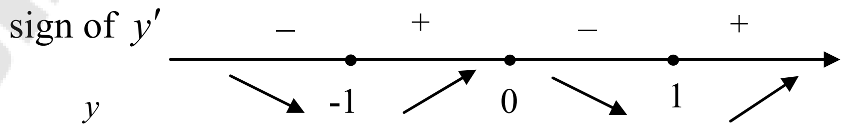
1. Calculate the derivative:

$$y' = \left(x^2 + \frac{1}{x^2}\right)' = \left(x^2 + x^{-2}\right)' = 2x - 2x^{-3} = 2x - \frac{2}{x^3} = \frac{2x^4 - 2}{x^3}.$$

2. Find the critical points:

$$\left[\begin{aligned} &2x^4 - 2 = 0, \\ &x^3 = 0 \end{aligned}\right. \Leftrightarrow \left[\begin{aligned} &x^4 = 1, \\ &x = 0 \end{aligned}\right. \Leftrightarrow \left[\begin{aligned} &x_{1,2} = \pm 1, \\ &x_3 = 0. \end{aligned}\right.$$

3. Determine the sign of the derivative:



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Function decreases with $x \in (-\infty; -1) \cup (0; 1)$.

Function decreases with $x \in (-1; 0) \cup (1; +\infty)$..

Points $x = 1$ and $x = -1$ are local minimums (by the first derivative test); point $x = 0$ is not local maximum because function $y = x^2 + \dfrac{4}{x^2}$ isn't defined at this point.

Example 2.9 . Determine if $x = 0$ is the extrema point of the function $y = \cos 2x + \operatorname{ch} x$.

Solution

Check at first the necessary condition:

$$y' = -2\sin 2x + \operatorname{sh} x; \quad y'(0) = -2\sin 0 + \operatorname{sh} 0 = 0.$$

So $x = 0$ is a critical point. However, it is difficult to determine the sign of the derivative to the right and to the left of the point $x = 0$.

Use the High Order Derivative Test. Find the second derivative of the function $y''(0)$:

$$y''(0) = \left(-4\cos 2x + \operatorname{ch} x\right)\Big|_{x=0} = -4 + 1 = -3.$$

Since the second derivative is negative at the point $x = 0$, then the function has a local maximum at this point.

II. Concavity.

Given the function $y = f(x)$ then

– $f(x)$ is ***concave up*** on an interval $(a; b)$ if all of the tangents to the curve on this interval are *below* the graph of $f(x)$ (see Fig. 2.2);

– $f(x)$ is ***concave down*** on an interval $(a; b)$ if all of the tangents to the curve on this interval are *above* the graph of $f(x)$ (see Fig. 2.3).

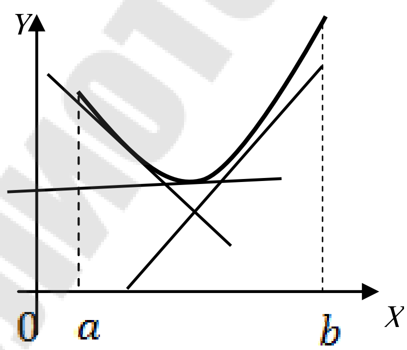

Fig. 2.2

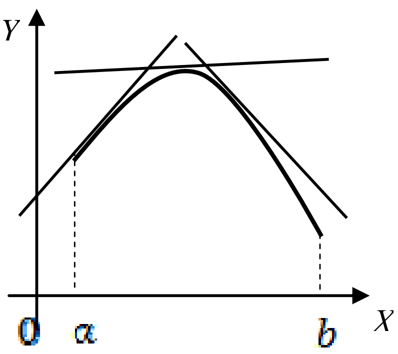

Fig. 2.3

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Conditions of conc avity up and down. Given the function $f(x)$ then

– if $f''(x) > 0$ for all x in some interval $(a;b)$ then $f(x)$ is concave up on this interval;

– if $f''(x) < 0$ for all x in some interval $(a;b)$ then $f(x)$ is concave down on this interval.

A point $x = c$ is called an ***inflection point*** if the function is continuous in it and the concavity of the graph changes at that point.

Sufficient condition for the presence of an inflection point . If $f''(c) = 0$ or does not exist and when passing through $x = c$ the derivative $f''(x)$ changes sign then $x = c$ is the inflection point.

Example 2.10. For the function $y = x^6 - 6x^5 + 7{,}5x^4 + 3x$ find the inflection points and the intervals of concave up/concave down.

Solution

One have to find the second derivative of given function:

$$y' = \left(x^6 - 6x^5 + 7{,}5x^4 + 3x\right)' = 6x^5 - 30x^4 + 30x^3 + 3;$$

$$y'' = \left(6x^5 - 30x^4 + 30x^3 + 3\right)' = 30x^4 - 120x^3 + 90x^2.$$

Solve the equation $y'' = 0$:

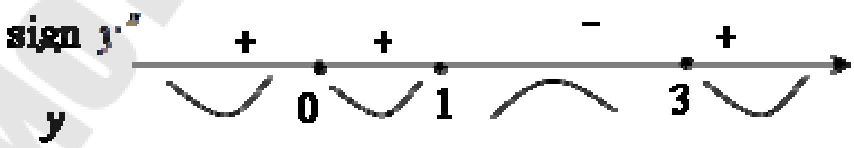
$$30x^4 - 120x^3 + 90x^2 = 0;$$

$$30x^2(x-1)(x-3) = 0;$$

$$x_1 = 0,\ x_2 = 1,\ x_3 = 3.$$

All these points belong to the domain of the function.

Investigate the sign of the second derivative on both sides of these points:



The derivative y'' changes sign passing through the points $x_2 = 1$ and $x_3 = 3$ so they both are inflection points. The derivative y'' doesn't change sign passing through the point $x_1 = 0$ so it is not inflection point.

Exercises

In the exercises 154–159 determine the intervals of monotonicity of the function, indicate the type of the extrema points:

154. $y = x\left(1 + 2\sqrt{x}\right).$ 155. $y = -\frac{x^2}{2} + 2x + \frac{8}{x-2} + 5.$

156. $y = 2\sqrt[3]{x^5} - 5\sqrt[3]{x^2} + 3.$ 157. $y = \frac{3x^2 + 4x + 4}{x^2 + x + 1}.$

158. $y = (x-2)^5(2x+1)^4.$ 159. $y = 2x^2 - \ln x.$

In the exercises 160–163, determine the intervals of concavity, find the inflection points:

160. $y = 3x^5 - 5x^4 + 3.$ 161. $y = (x+2)^5 - 2x + 2.$

162. $y = \ln(1 + x^2).$ 163. $y = xe^{2x} + 1.$

In the exercises 164–169, conduct a complete study of functions and build graphs:

164. $y = 32x^2(x^2-1)^3.$ 165. $y = \frac{x}{x^2 - 1}.$

166. $y = \frac{x^2 + 2x - 1}{2x + 1}.$ 167. $y = \frac{8(x-1)}{(x+2)^2}.$

168. $y = x^2 e^{-x}.$ 169. $y = (2x+3)e^{-2(x+1)}.$

SECTION 3. FUNCTIONS OF SEVERAL VARIABLES

3.1. Basic Concepts

Let D be the set of ordered pairs of numbers $(x; y)$. They say that a function $z = f(x, y)$ of two variables is given on the set D if the law is known according to which a certain number z is assigned to each pair $(x; y)$ belonging to D.

The *Domain* of function of two variables $z = f(x, y)$ is a regions from two dimensional space, it consists of all the coordinate pairs $(x; y)$ that we could plug into the function and get back a real number. The set of the actual values produced by the function is called the *Range*.

Example 3.1. Find domain and range of a function $z = \sqrt{1 - x^2 - y^2}$.

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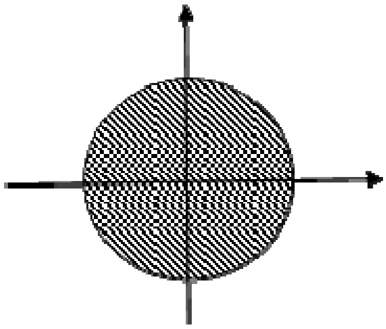
Solution

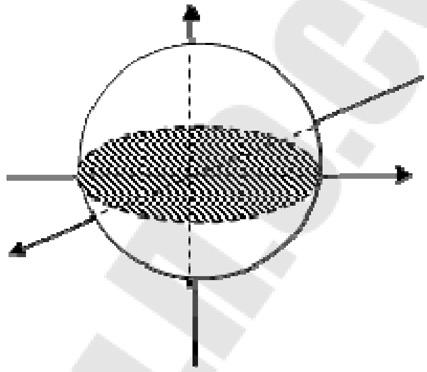
Domain: $1-x^2-y^2 \geq 0.$

$x^2+y^2 \leq 1$ – circle with radius 1:



Range: $z^2 = 1-x^2-y^2.$

$x^2+y^2+z^2 = 1$ – sphere with radius 1:



Similarly, you can define a function of three or more variables.

Let's remember the definition of the limit in the case of function of one variable $f(x)$:

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a-0} f(x) = \lim_{x\to a+0} f(x) = L.$$

In the case of $f(x,y)$ the ***limit*** is written as

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Limit $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y)$ exists if L is independent on the way $(x,y)\to(a,b)$.

A function $f(x,y)$ is ***continuous*** at the point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

Example 3.2. Determine if the limit $\lim\limits_{(x,y)\to(1,1)} \dfrac{2x^2-xy-y^2}{x-y}$ exists or not. If it does exist give the value of the limit.

Solution

$$\lim_{(x,y)\to(1,1)} \frac{2x^2-xy-y^2}{x-y} = \left\{\frac{2\cdot 1^2 - 1\cdot 1 - 1^2}{1-1} = \frac{0}{0}\right\}.$$

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Factorize the numerator:

$$2x^2 - xy - y^2 = x^2 - xy + x^2 - y^2 = x(x-y) + (x-y)(x+y) =$$
$$= (x-y)\big(x + (x+y)\big) = (x-y)(2x+y).$$

Then

$$\lim_{(x,y)\to(1,1)} \frac{2x^2 - xy - y^2}{x-y} = \lim_{(x,y)\to(1,1)} \frac{(x-y)(2x+y)}{x-y} =$$
$$= \lim_{(x,y)\to(1,1)} (2x+y) = 2+1 = 3.$$

3.2. Partial Derivatives

Let $z = f(x,y)$ and let some arbitrary point (x_0, y_0) is fixed.

The change in the function $f(x,y)$ associated with the change in the variable x with a fixed value of y is called the *partial increment of the function* $z = f(x,y)$ *in respect to* x:

$$\Delta_x z = f(x_0 + \Delta x, y_0) - f(x_0, y_0).$$

Definition. The ***partial derivative*** for function $f(x,y)$ ***in respect to*** x is defined to be

$$\lim_{\Delta x\to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta_x z}{\Delta x} = f_x(x,y).$$

Other notations that can be used are

$$f_x(x,y) = \frac{\partial f}{\partial x} = z'_x.$$

The *partial increment* $\Delta_y z$ *of the function* $f(x,y)$ and the ***partial derivative*** $f_y(x,y)$ *in respect to* y are defined in a similar way.

Note that when one takes the partial derivative, e.g., with respect to x, it is necessary to hold the other variables as constants. So, partial derivatives have the same properties as ordinary derivative as well as all rules of differentiation hold.

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Examples 3.3. Find all of the first order partial derivatives for the function $f(x,y) = 2xy^3 + x^2 \ln y + e^{2x}$.

Solution

To find partial derivative with respect to x we hold y as a constant:

$$f_x = \left(2xy^3 + x^2 \ln y + e^{2x}\right)_x' = y^3(2x)_x' + \ln y\left(x^2\right)_x' + \left(e^{2x}\right)_x' =$$

$$= 2y^3 + 2x\ln y + 2e^{2x}.$$

To find partial derivative with respect to y we hold x as a constant:

$$f_y = \left(2xy^3 + x^2 \ln y + e^{2x}\right)_y' = 2x\left(y^3\right)_y' + x^2(\ln y)_y' + \left(e^{2x}\right)_y' =$$

$$= 2x \cdot 3y^2 + x^2\frac{1}{y} + 0 = 6xy^2 + \frac{x^2}{y}.$$

The expression

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

is called the ***total differential*** of the function $z = f(x,y)$.

Exercises

In the exercises 170, 171 find the domain of given functions:

170. $z = \dfrac{1}{\sqrt{x+y}} + \sqrt{x-y}.$

171. $z = \arcsin(x-y).$

In the exercises 172–179 find all the partial derivatives and total differential for each of the following functions:

172. $z = 4x^3y^2 - xy + 3x - 5.$

173. $z = x\sqrt{y} + \dfrac{y}{\sqrt{x}}.$

174. $y = \arcsin\left(\dfrac{x}{y}\right).$

175. $z = e^{2x-3y} + 3xy^4 + y.$

176. $z = x\sin 2y + 3y^2 - \dfrac{x}{y}.$

177. $z = \sqrt{x^2 - 2y} + 2x^2y + 4.$

178. $z = \dfrac{1}{x^2 + y^2} - 5x^2y^3.$

179. $u = x^2yz - xy^z + \sqrt{z}.$

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3.3. Higher Order Partial Derivatives

Consider the case of a function of two variables, $z = f(x, y)$ since both of the first order partial derivatives are also functions of x and y:

$$f_x = \frac{\partial f}{\partial x} = f_x(x, y), \qquad f_y = \frac{\partial f}{\partial y} = f_y(x, y).$$

This means that for the case of a function of two variables there will be a total of four possible second order derivatives:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}; \qquad (f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x};$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}; \qquad (f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}.$$

Example 3.4. Find all the second order derivatives for

$$f(x, y) = \cos 2x + x^2 e^{3y} - 2y^3.$$

Solution

First of all one have to find first partial derivatives:

$$f_x(x, y) = \left(\cos 2x + x^2 e^{3y} - 2y^3\right)_x' = (\cos 2x)_x' + \left(x^2 e^{3y}\right)_x' - \left(2y^3\right)_x' =$$

$$= -2\sin 2x + +2xe^{3y};$$

$$f_y(x, y) = \left(\cos 2x + x^2 e^{3y} - 2y^3\right)_y' = (\cos 2x)_y' + \left(x^2 e^{3y}\right)_y' - \left(2y^3\right)_y' =$$

$$= 3x^2 e^{3y} - 6y^2.$$

Second order partial derivatives:

$$f_{xx} = \left(-2\sin 2x + +2xe^{3y}\right)_x' = -4\cos 2x + 2e^{3y};$$

$$f_{xy} = \left(-2\sin 2x + +2xe^{3y}\right)_y' = 6xe^{3y};$$

$$f_{yx} = \left(3x^2 e^{3y} - 6y^2\right)_x' = 6xe^{3y};$$

$$f_{yy} = \left(3x^2 e^{3y} - 6y^2\right)_x' = 9x^2 e^{3y} - 12y.$$

One can note $f_{xy} = f_{yx}$ and this is no accident.

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Clairaut's Theorem. Suppose that f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on this disk then, $f_{xy}(a,b) = f_{yx}(b,a)$.

This statement is true for any high order mixed derivatives.

3.4. The Extrema of Function of Several Variables

Definition. A function $z = f(x, y)$ has a *local minimum* at the point $M(a,b)$ if $f(x, y) \geq f(a,b)$ for all points (x, y) in some region around $M(a,b)$. A function $f(x, y)$ has a *local maximum* at the point $M(a,b)$ if $f(x, y) \leq f(a,b)$ for all points (x, y) in some region around $M(a,b)$.

The Necessary Condition for the Extremum of a Function of Two Variables. If the point $M(a,b)$ is a local extrema of the function $f(x, y)$ then at this point either both of its partial derivatives f_{xy} and f_{yx} are equal to zero, or at least one of them does not exist.

Points satisfying the necessary condition are called *critical points*.

Note that this does NOT say that all critical points are local extrema!

The Sufficient Condition for the Extremum of a Function of Two Variables . Suppose that $M(a,b)$ is a critical point of the function $z = f(x, y)$ and that the second order partial derivatives are continuous in some region that contains $M(a,b)$. Next define

$$A = f''_{xx}(a,b), \quad B = f''_{yy}(a,b), \quad C = f''_{xy}(a,b),$$

$$\Delta = AB - C^2.$$

We have the following classifications of the critical points:

– $\Delta > 0$ and $A > 0$ then there is a ***local minimum*** at $M(a,b)$;

– $\Delta > 0$ and $A < 0$ then there is a ***local maximum*** at $M(a,b)$;

– $\Delta < 0$ then the point $M(a,b)$ is a saddle point;

– $\Delta = 0$ then the point $M(a,b)$ may be a relative minimum, relative maximum or a saddle point. Other techniques would need to be used to classify the critical point.

Example 3.5. Find and classify all the critical points of the function $z = x^3 + y^3 - 6xy$.

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Solution

a) We need all the first order partial derivatives to find the critical points:

$$z'_x = \left(x^3 + y^3 - 6xy\right)'_x = 3x^2 - 6y;$$

$$z'_y = \left(x^3 + y^3 - 6xy\right)'_y = 3y^2 - 6x.$$

b) Critical points will be solutions to the following system:

$$\begin{cases} 3x^2 - 6y = 0, \\ 3y^2 - 6x = 0 \end{cases} \Rightarrow \begin{cases} x^2 - 2y = 0, \\ y^2 - 2x = 0 \end{cases} \Rightarrow \begin{cases} y = \dfrac{x^2}{2}, \\ y^2 - 2x = 0. \end{cases}$$

$$\left(\frac{x^2}{2}\right)^2 - 2x = 0;$$

$$\frac{x^4}{4} - 2x = 0;$$

$$x^4 - 8x = 0,\ x(x^3 - 8) = 0;$$

$$x_1 = 0,\ x_2 = 2.$$

After substitution x_1 and x_2 into the equation $y = \dfrac{x^2}{2}$ we get $y_1 = 0,\ y_2 = 2.$ So, we have two critical points: $M_1(0,0)$ and $M_2(2,2)$.

c) We need all the second order partial derivatives to classify the critical points:

$$z''_{xx} = (3x^2 - 6y)'_x = 6x;$$

$$z''_{xy} = (3x^2 - 6y)'_y = -6;$$

$$z''_{yy} = (3y^2 - 6x)'_y = 6y.$$

For the critical point $M_1(0,0)$ we have:

$$A = z''_{xx}(0,0) = 0,\ B = z''_{xy}(0,0) = -6,\ C = z''_{yy}(a, b) = 0;$$

$$\Delta = AC - B^2 = 0 - (-6)^2 = 36 < 0,$$

so critical point $M_1(0,0)$ is a saddle point.

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For the critical point $M_2(2,2)$ we have:

$$A = z''_{xx}(0,0) = 6\cdot 2 = 12;\ B = z''_{xy}(0,0) = -6;\ C = z''_{yy}(a,b) = 12;$$

$$\Delta = AC - B^2 = 12\cdot 12 - (-6)^2 = 108 > 0.$$

$\Delta > 0,\ A > 0$ so function has local minimum at a point $M_2(2,2)$.

Exercises

In the exercises 180, 181 find the second order derivatives:

180. $z = e^{x^2 - y^2}$. 181. $z = \sin\sqrt{xy}$.

In the exercises 182 – 185 find and classify all the critical points of given functions:

182. $z = x^2 - xy + y^2 + 6x - 6y + 5$. 183. $z = 4(x-y) - x^2 - y^2$.

184. $z = x^2 - xy + y^2 + 6x - 6y + 5$. 185. $z = xy(12 - x - y)$.

SECTION 4. INDEFINITE INTEGRAL

4.1. Basic Concepts

Definition. Given a function, $f(x)$, an anti-derivative of $f(x)$ is any function $F(x)$ such that

$$F'(x) = f(x).$$

Consider the function $f(x) = 2x$. Obviously, function $F_1(x) = x^2$ is anti-derivative of $f(x) = 2x$ because $F_1'(x) = \left(x^2\right)' = 2x = f(x)$. But $F_1(x)$ is not a unique anti-derivative of $f(x)$. For example, $\left(x^2 + 5\right)' = 2x = f(x)$, so the function $F_2(x) = x^2 + 5$ is also anti-derivative of $f(x)$. We can conclude that if $F(x)$ is the anti-derivative of a function $f(x)$ on the interval $(a;b)$ then any function $F_1(x) = F(x) + C$ is also the anti-derivative for $f(x)$ on $(a;b)$.

Definition. If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an *indefinite integral* and denoted

$$\int f(x)dx = F(x) + C,$$

where C is a constant of integration.

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The process of finding the indefinite integral is called *integration* or *integrating* $f(x)$.

Properties of Indefinite Integral:

1) $\left(\int f(x)dx\right)' = f(x)$;

2) $\int f'(x)dx = f(x) + C$;

3) $d\int f(x)dx = f(x)dx$;

4) $\int df(x) = f(x) + C$;

5) $\int kf(x)dx = k\int f(x)dx$;

6) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$;

7) $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$.

According to property 1, the integration operation is inverse to the differentiation operation. We will use the table to calculate the indefinite integrals.

List of main indefinite integrals:

1. $\int dx = x + C$.

2. $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1)$.

3. $\int \frac{dx}{x-a} = \ln|x-a| + C$.

4. $\int a^x dx = \frac{a^x}{\ln a} + C$.

5. $\int e^x dx = e^x + C$.

6. $\int \cos x dx = \sin x + C$.

7. $\int \sin x dx = -\cos x + C$.

8. $\int \frac{dx}{\cos^2 x} = \mathrm{tg}\, x + C$.

9. $\int \frac{dx}{\sin^2 x} = -\mathrm{ctg}\, x + C$.

10. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\mathrm{arctg}\frac{x}{a} + C = -\frac{1}{a}\mathrm{arcctg}\frac{x}{a} + C$.

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11. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{a-x}{a+x}\right| + C = -\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C.$

12. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a}\right| + C \quad (a \neq 0).$

13. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C = -\arccos\frac{x}{a} + C.$

14. $\int \operatorname{sh} x dx = \operatorname{ch} x + C.$

15. $\int \operatorname{ch} x dx = \operatorname{sh} x + C.$

16. $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C.$

17. $\int \frac{dx}{sh^2 x} = -\operatorname{cth} x + C.$

4.2. Direct Integration Method

The method of direct integration consists in reducing a given integral to the sum or difference of main integrals by means of identical transformations of the integrand.

Example 4.1. Evaluate each of the following integrals:

a) $\int \left(x^4 + 2x^2 + 3x + 5\right)dx;$ b) $\int \left(5\sqrt[3]{x^2} - \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\right)dx;$

c) $\int \frac{\left(2x^2 + 1\right)^2}{\sqrt{x}}dx;$ d) $\int \left(\cos x \cdot \operatorname{tg} x - 5^x\right)dx.$

Solution

a) $\int \left(x^4 + 2x^2 + 3x + 5\right)dx = \{\text{properties 6 and 5}\} =$

$$= \int x^4 dx + 2\int x^2 dx + 3\int x dx + 5\int dx = \left\{\begin{matrix}\text{1 and 2 from} \\ \text{the list of integrals}\end{matrix}\right\} =$$

$$= \frac{x^{4+1}}{4+1} + 2\frac{x^{2+1}}{2+1} + 3\frac{x^{1+1}}{1+1} + 5x + C = \frac{x^5}{5} + \frac{2x^3}{3} + \frac{3x^2}{2} + 5x + C.$$

b) One have to rewrite the integrand as a sum of power functions:

$$\sqrt[3]{x^2} = x^{\frac{2}{3}}; \quad \frac{1}{x^5} = x^{-5}; \quad \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}};$$

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$$\int\left(5\sqrt[3]{x^2}-\frac{7}{x^5}+\frac{1}{6\sqrt{x}}\right)dx=\int\left(5x^{2/3}-7x^{-5}+\frac{1}{6}x^{-1/2}\right)dx=$$

$$=\{\text{properties 6 and 5}\}=5\int x^{2/3}dx-7\int x^{-5}dx+\frac{1}{6}\int x^{-1/2}dx=$$

$$=\left\{\begin{matrix}\text{2 from the list}\\ \text{of integrals}\end{matrix}\right\}=5\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}-7\frac{x^{-5+1}}{-5+1}+\frac{1}{6}\cdot\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C=$$

$$=3x^{\frac{5}{3}}+\frac{7}{4}x^{-4}+\frac{1}{3}x^{\frac{1}{2}}+C=3x\sqrt[3]{x^2}+\frac{7}{4x^4}+\frac{\sqrt{x}}{3}+C.$$

c) There is no rule for dealing with quotients. In this case all we need to do is break up the quotient and then integrate the individual terms:

$$\int\frac{(2x^2+1)^2}{\sqrt{x}}dx=\left\{\begin{matrix}(a+b)^2=a^2+2ab+b^2\\ (2x^2+1)^2=4x^4+4x^2+1\end{matrix}\right\}=\int\frac{4x^4+4x^2+1}{\sqrt{x}}dx=$$

$$=\int\left(\frac{4x^4}{x^{1/2}}+\frac{4x^2}{x^{1/2}}+\frac{1}{x^{1/2}}\right)dx=4\int x^{\frac{7}{2}}dx+4\int x^{\frac{3}{2}}dx+\int x^{-\frac{1}{2}}dx=$$

$$=\left\{\begin{matrix}\text{2 from the list}\\ \text{of integrals}\end{matrix}\right\}=4\frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1}+4\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}+\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C=$$

$$=4\frac{x^{\frac{9}{2}}}{\frac{9}{2}}+4\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+C=\frac{8x^4\sqrt{x}}{9}+\frac{8x^2\sqrt{x}}{5}+2\sqrt{x}+C=$$

$$=\frac{2\sqrt{x}}{45}\left(20x^4+36x^2+45\right)+C.$$

d) $\int\left(\cos x\cdot\operatorname{tg}x-5^x\right)dx=\{\text{property 6}\}=\int\cos x\cdot\frac{\sin x}{\cos x}dx-\int 5^x dx=$

$$=\int\sin x dx-\int 5^x dx=\left\{\begin{matrix}\text{4 and 7 from}\\ \text{the list of integrals}\end{matrix}\right\}=-\sin x-\frac{5^x}{\ln 5}+C.$$

Example 4.2. Evaluate each of the following integrals:

a) $\int \frac{3}{x^2+4}dx$; b) $\int \frac{dx}{4x^2-9}$; c) $\int \frac{dx}{\cos^2(3x+5)}$.

Solution

a) $$\int \frac{3}{x^2+4}dx = \{\text{property } 5\} = 3\int \frac{dx}{x^2+4} = 3\int \frac{dx}{x^2+2^2} =$$

$$= \begin{Bmatrix} \text{10 from the list} \\ \text{of integrals} \end{Bmatrix} = 3\cdot\frac{1}{2}\operatorname{arctg}\frac{x}{2} + C.$$

b) $$\int \frac{dx}{4x^2-9} = \int \frac{dx}{4\left(x^2-\frac{9}{4}\right)} = \frac{1}{4}\int \frac{dx}{x^2-\left(\frac{3}{2}\right)^2} = \begin{Bmatrix} \text{11 from the list} \\ \text{of integrals} \end{Bmatrix} =$$

$$= \frac{1}{4}\cdot\frac{1}{2\cdot\frac{3}{2}}\ln\left|\frac{x-\frac{3}{2}}{x+\frac{3}{2}}\right| + C = \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C.$$

c) $$\int \frac{dx}{\cos^2(3x+5)} = \begin{Bmatrix} \text{property 7;} \\ \text{8 from the list of integrals} \end{Bmatrix} = \frac{1}{3}\operatorname{tg}(3x+5) + C.$$

Exercises

In the exercises 185–202 evaluate the following integrals:

185. $\int (2x^3 - 7x + 4)dx$.

186. $\int (3x^3 - 5)^2 dx$.

187. $\int \frac{3x^2 - 4x + 5}{x}dx$.

188. $\int \left(\frac{1}{2x} + \sqrt[7]{x^5} + \frac{5}{\sqrt[3]{x}}\right)dx$.

189. $\int \left(\frac{1}{x} - 3\sqrt{x}\right)^2 dx$.

190. $\int \sqrt{x}\left(3x + \sqrt[4]{x}\right)dx$.

191. $\int \frac{x\sqrt{x} + \sqrt[3]{x} + 1}{x^2}dx$.

192. $\int \frac{dx}{x^2 - 25}$.

193. $\int \frac{4dx}{\sqrt{3 - x^2}}$.

194. $\int \frac{dx}{\sqrt{6 + 5x^2}}$.

195. $\int \frac{dx}{8x^2 + 16}$.

196. $\int \frac{dx}{3x - 2}$.

197. $\int e^{4x}dx$.

198. $\int (\sin 2x - \cos 2x)dx$.

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199. $\int \frac{dx}{5^{4-3x}}$.

200. $\int \cos(4-7x)dx$.

201. $\int \frac{dx}{\sin^2 5x}$.

202. $\int \frac{dx}{2-x}$.

4.3. Substitution Rule

The technique of substitutions helps to reduce integral to common indefinite integrals, which are given in list of main integrals by means of introducing new variables in the form $u = \psi(x)$ or $x = \varphi(t)$.

A natural question is how to identify the correct substitution. Unfortunately, the answer depends on the integral. But there is a set of standard cases in which such a substitutions are correct and useful.

As a rule, the substitution $u = \psi(x)$ is used when a given integral has the following structure:

$$\int f\left(\psi(x)\right)\psi'(x)\,dx.$$

Example 4.3. Evaluate each of the following integrals:

a) $\int x \sin x^2 dx$; b) $\int \frac{\ln^3 x}{x} dx$.

Solution

a) $$\int x \sin x^2 dx = \begin{Bmatrix} u = x^2 \\ du = 2xdx \Rightarrow xdx = \frac{1}{2}du \end{Bmatrix} = \frac{1}{2}\int \sin u\, du =$$

$$= \begin{Bmatrix} \text{7 from the list} \\ \text{of integrals} \end{Bmatrix} = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos x^2 + C.$$

b) $$\int \frac{\ln^3 x}{x} dx = \begin{Bmatrix} u = \ln x \\ du = \frac{dx}{x} \end{Bmatrix} = \int u^3 du = \begin{Bmatrix} \text{2 from the list} \\ \text{of integrals} \end{Bmatrix} = \frac{u^4}{4} + C =$$

$$= \frac{\ln^4 x}{4} + C.$$

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Exercises

In the exercises 203–220 evaluate the following integrals:

203. $\int \frac{xdx}{1+x^2}$. 204. $\int x^2\left(2x^3+5\right)^{10}dx$. 205. $\int \frac{\ln x}{x}dx$.

206. $\int x\sqrt{x^2+8}dx$. 207. $\int \sin^2 x\cos xdx$. 208. $\int \frac{2xdx}{\sqrt{3x^2-1}}$.

209. $\int \frac{\sin 3x}{\cos^2 3x}dx$. 210. $\int \frac{\ln^3(x+2)}{x+2}dx$. 211. $\int \operatorname{tg} xdx$.

212. $\int \frac{\operatorname{tg}^6 x}{\cos^2 x}dx$. 213. $\int \frac{\sin x}{2+3\cos x}dx$. 214. $\int \frac{\sqrt{\operatorname{arctg} x}}{1+x^2}dx$.

215. $\int \frac{e^x dx}{e^x+3}$. 216. $\int \frac{dx}{x\ln x}$. 217. $\int \frac{3x-2}{x^2-9}dx$.

218. $\int \frac{3-4\operatorname{ctg} x}{\sin^2 x}dx$. 219. $\int \sqrt{\frac{\arcsin 2x}{1-4x^2}}dx$. 220. $\int \frac{e^x dx}{e^{2x}+1}$.

4.4. Integration by Parts

The method of integration by parts is based on the application of the formula

$$\int udv = u\cdot v - \int vdu, \tag{4.1}$$

where $u(x)$ and $v(x)$ are continuously differentiable functions at some interval.

To use this formula, we will need:

– to identify u and dv;

– compute du: $du = u'(x)dx$;

– compute v: $v = \int dv$.

How do we know if we made the correct choice for u and dv?

We made the correct choices for u and dv if, after using the integration by parts formula, the new integral (the one on the right of the formula) is one we can actually integrate.

There are a number of standard cases where the integration by parts formula should be applied:

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a) $\int P_n(x)e^{ax}dx$, $\int P_n(x)b^{ax}dx$, $\int P_n(x)\cos axdx$, $\int P_n(x)\sin axdx$, where $P_n(x)$ is a polynomial of the n-th degree of x.

In this case $P_n(x)$ should be chosen for $u(x)$ and $e^{ax}dx$ (correspondently, $b^{ax}dx$, or $\cos axdx$ or $\sin axdx$) for dv.

Example 4.4

$$\int \underbrace{x}_{u}\underbrace{e^{2x}dx}_{dv} = \left\{\begin{matrix} u = x, & du = dx \\ dv = e^{2x}dx, & v = \frac{1}{2}e^{2x} \\ \multicolumn{2}{c}{\int udv = u\cdot v - \int vdu} \end{matrix}\right\} = x\cdot\frac{1}{2}e^{2x} - \frac{1}{2}\int e^{2x}\cdot dx =$$

$$= x\cdot\frac{1}{2}e^{2x} - \frac{1}{2}\cdot\frac{1}{2}e^{2x} + C = \frac{1}{4}e^{2x}(2x-1) + C.$$

Note that after applying the formula (4.1), we come to an integral simpler in relation to the original. Formula of integration by parts can be applied several times, until we come to an integral that can be computed in elementary functions:

b) $\int P_n(x)\ln xdx$, $\int P_n(x)\arccos axdx$, $\int P_n(x)\arcsin axdx$, $\int P_n(x)\mathrm{arctg}\, axdx$, $\int P_n(x)\mathrm{arcctg}\, axdx$.

In this case, for dv you should choose $P_n(x)dx$.

Example 4.5

$$\int x\ln xdx = \left\{\begin{matrix} u = \ln x, & du = (\ln x)'dx = \frac{1}{x}dx \\ dv = xdx, & v = \frac{x^2}{2} \\ \multicolumn{2}{c}{\int udv = u\cdot v - \int vdu} \end{matrix}\right\} =$$

$$= \ln x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int xdx = \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + C =$$

$$= \frac{x^2}{4}(2\ln x - 1) + C.$$

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Exercises

In the exercises 221–228 evaluate the following integrals:

221. $\int x\sin 2x dx.$ 222. $\int (3x+1)e^{2x}dx.$

223. $\int (x+1)\cos(4x-1)dx.$ 224. $\int \ln(2x-1)dx.$

225. $\int (x^2+1)\ln x dx.$ 226. $\int \frac{\ln x}{\sqrt{x^3}}dx.$

227. $\int \arcsin x dx.$ 228. $\int \cos\sqrt{x}dx.$

4.5. Integrals Involving Quadratics

Type 1. The evaluation of integrals of the form

$$I=\int \frac{dx}{x^2+px+q}$$

consists in reducing this integral to one of the tabular integrals of type 2, 10, and 11.

One have to complete the square on the denominator of the integrand:

$$x^2+px+q=\left[x^2+2x\cdot\frac{p}{2}+\left(\frac{p}{2}\right)^2\right]-\left(\frac{p}{2}\right)^2+q=\left(x+\frac{p}{2}\right)^2-\frac{p^2-4q}{4}.$$

Given integral can be simplified by substitution:

$$x+\frac{p}{2}=t,\quad dx=dt.$$

Example 4.6. Evaluate integral $\int \frac{dx}{x^2+4x+5}$.

Solution

Let's complete the full square in the denominator of the integrand using formula:

$$a^2\pm 2ab+b^2=(a\pm b)^2.$$

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We get

$$x^2+4x+5=\left(x^2+2x\cdot 2+2^2\right)-2^2+5=(x+2)^2+1.$$

Then

$$\int\frac{dx}{x^2+4x+5}=\int\frac{dx}{(x+2)^2+1}=\left\{\begin{matrix}x+2=t\\ dx=dt\end{matrix}\right\}=\int\frac{dx}{t^2+1}=$$

$$=\left\{\begin{matrix}\text{10 from the list}\\ \text{of integrals}\end{matrix}\right\}=\operatorname{arctg} t+C=\operatorname{arctg}(x+2)+C.$$

Type 2. To evaluate the integral of the form

$$I=\int\frac{mx+n}{x^2+px+q}dx$$

one have to complete the square in the denominator of the integrand, do standard substitution $x+\frac{p}{2}=t$ and separate the integral into a sum of two, one of which is calculated by the formula (3) and the second by formula (10) or 11 from the list of integrals.

Example 4.6. Evaluate integral $\int\frac{4x+1}{x^2-2x-3}dx.$

Solution

Complete the full square in the denominator:

$$x^2-2x-3=\left(x^2-2x\cdot 1+1^2\right)-1^2-3=(x-1)^2-4.$$

Then

$$\int\frac{4x+1}{x^2-2x-3}dx=\int\frac{4x+1}{(x-1)^2-4}dx=\left\{\begin{matrix}x-1=t\\ x=t+1\\ dx=dt\end{matrix}\right\}=\int\frac{4(t+1)+1}{t^2-4}dt=$$

$$=\int\frac{4t+5}{t^2-4}dt=4\int\frac{tdt}{t^2-4}+5\int\frac{dt}{t^2-4}=4I_1+5I_2.$$

Note that the integral I_1 can be calculated by substitution $y=t^2-4$, integral I_2 one can find in a table of integrals (formula number 11).

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$$I_1=\int\frac{tdt}{t^2-4}=\left\{\begin{matrix} y=t^2-4 \\ dy=2tdt\end{matrix}\right\}=\frac{1}{2}\int\frac{dy}{y}=\left\{\begin{matrix}\text{10 from the list} \\ \text{of integrals}\end{matrix}\right\}=$$

$$=\frac{1}{2}\ln|y|=\frac{1}{2}\ln|t^2-4|=\frac{1}{2}\ln|x^2-2x-3|.$$

$$I_2=\int\frac{dt}{t^2-4}=\int\frac{dt}{t^2-2^2}=\left\{\begin{matrix}\text{11 from the list} \\ \text{of integrals}\end{matrix}\right\}=\frac{1}{2\cdot 2}\ln\left|\frac{t-2}{t+2}\right|=$$

$$=\frac{1}{4}\ln\left|\frac{(x-1)-2}{(x-1)+2}\right|=\frac{1}{4}\ln\left|\frac{x-3}{x+1}\right|.$$

Finally obtain:

$$\int\frac{4x+1}{x^2-2x-3}dx=4I_1+5I=2\ln|x^2-2x-3|+\frac{5}{4}\ln\left|\frac{x-3}{x+1}\right|+C.$$

Remark. This method can be also used for evaluating integrals of the form

$$\int\frac{dx}{\sqrt{\pm x^2+px+q}},\text{ or}\int\frac{mx+n}{\sqrt{\pm x^2+px+q}}dx.$$

Exercises

In the exercises 229–236 evaluate the following integrals:

229. $\int\frac{7dx}{-x^2-4x+1}$.

230. $\int\frac{3dx}{x^2+6x}$.

231. $\int\frac{dx}{\sqrt{x^2+2x+2}}$.

232. $\int\frac{dx}{\sqrt{3+2x-x^2}}$.

233. $\int\frac{x+2}{x^2+6x-8}dx$.

234. $\int\frac{2x+1}{2x^2+2x+9}dx$.

235. $\int\frac{x+1}{\sqrt{4x-x^2}}dx$.

236. $\int\frac{3x-2}{\sqrt{x^2+3x-5}}dx$.

4.6. Integration of Rational Functions

A rational function is a function that can be expressed as the ratio of two polynomials:

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$$R(x)=\frac{P_n(x)}{Q_m(m)}=\frac{a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\ldots+a_{m-1}x+b_m},\ a_0\neq 0,\ b_0\neq 0.$$

A rational function $\frac{P_n(x)}{Q_m(x)}$ is said to be a *proper fraction* if the degree of the polynomial in the numerator n is less than that in the denominator m.

Fractions of the following forms:

1. $\frac{A}{x-a}$.
2. $\frac{B}{(x-b)^k}$.
3. $\frac{Mx+N}{x^2+px+q}$ with $D=p^2-4q<0$.
4. $\frac{Kx+L}{(x^2+px+q)^l}$ with $D=p^2-4q<0$

are called the *partial fractions*.

Let $R(x)=\frac{P_n(x)}{Q_m(x)}$ be a proper fraction. The procedure of integration $R(x)$ consists of following steps:

– factorize denominator $Q_m(x)$ into irreducible polynomials, that are, linear and irreducible quadratic polynomials.

– for each factor in the denominator we can use the following table to determine the term(s) we pick up in *the partial fraction decomposition*:

Factor in denominator	Term in the partial fraction decomposition
$x-a$	$\frac{A}{x-a}$
$(x-a)^k$	$\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\cdots+\frac{B_k}{(x-b)^k}$
x^2+px+q	$\frac{Mx+N}{x^2+px+q}$
$(x^2+px+q)^l$	$\frac{K_1x+L_1}{x^2+px+q}+\frac{K_2x+L_2}{(x^2+px+q)^2}+\cdots+\frac{K_lx+L_l}{(x^2+px+q)^l}$

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Thus the rational function $R(x)$ can be represent as a sum of partial fractions with unknown coefficients which can be found using the *undeterminel coefficient method.* To do this, it is necessary to:

Step 1: bring to a common denominator the sum of partial fractions.

Step 2: equate the numerator of the resulting fraction to $P_n(x)$.

Step 3: equate the coefficients at the same degrees x in the equality obtained in a Step 2.

Step 4: solve the resulting system of linear equations with respect to unknown $A, B_1, \dots, B_k, M, N, K_1, \dots, K_l, L_1, \dots, L_l, \dots$.

Partial fractions (1)–(3) are easily integrated (see subsection 4.5). To calculate the fourth integral, the following recursive formula is useful:

$$J_l = \int \frac{dx}{\left(x^2 + px + q\right)^l} =$$

$$= \frac{2x + p}{(l-1)\left(4q - p^2\right)\left(x^2 + px + q\right)^{l-1}} + \frac{2l-3}{l-1} \cdot \frac{2}{4q - p^2} \cdot J_{l-1}.$$

Remark. If $R(x)$ is improper fraction it is necessary at first to perform the polynomial long division in order to represent the function $R(x)$ as a sum of some polynomial and the remainder term (which is a proper fraction).

Example 4.7 . Evaluate the integral $\int \frac{7x-11}{(x-2)(x^2-x-2)} dx.$

Solution

Integrand is a proper rational function so we have to factorize the denominator into irreducible factors. For

$$x^2 - x - 2$$

$D = (-1)^2 - 4 \cdot (-2) = 9 > 0$, so it can be factorized as

$$x^2 - x - 2 = (x+1)(x-2).$$

So the partial fraction decomposition of integrand is

$$\frac{4x-5}{(x-2)(x^2-x-2)} = \frac{4x-5}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B_1}{x-2} + \frac{B_2}{(x-2)^2}.$$

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Find undetermined coefficients A, B_1, B_2. Bring the sum of partial fractions to a common denominator:

$$\frac{4x-5}{(x+1)(x-2)^2}=\frac{A}{x+1}+\frac{B_1}{x-2}+\frac{B_2}{(x-2)^2}=$$

$$=\frac{A(x-2)^2+B_1(x+1)(x-2)+B_2(x+1)}{(x+1)(x-2)^2}.$$

Let's equate the numerators of the resulting and the original fractions:

$$A(x-2)^2+B_1(x+1)(x-2)+B_2(x+1)=7x-11;$$

$$A(x-4x+4)+B_1(x^2-x-2)+B_2(x+1)=7x-11;$$

$$x^2(A+B_1)+x(-4A-B_1+B_2)+(4A-2B_1+B_2)=7x-11.$$

Let's equate the coefficients at the same degrees x:

$$\left.\begin{array}{ll} x^2: & A+B_1=0, \\ x^1: & -4A-B_1+B_2=7, \\ x^0: & 4A-2B_1+B_2=-11. \end{array}\right\}$$

System solution: $A=-2,\ B_1=2,\ B_2=1.$

Substitute the resulting decomposition into an integral:

$$\int\frac{7x-11}{(x-2)(x^2-x-2)}dx=\int\left(\frac{-2}{x+1}+\frac{2}{x-2}+\frac{1}{(x-2)^2}\right)dx=$$

$$=-2\int\frac{dx}{x+1}+2\int\frac{dx}{x-2}+\int\frac{dx}{(x-2)^2}=\left\{\begin{array}{l}\text{2, 3 from the list}\\ \text{of integrals}\end{array}\right\}=$$

$$=-2\ln|x+1|+2\ln|x-2|-\frac{1}{x-2}+C.$$

Example 4.8. Evaluate the integral $\int\frac{2x^5+6x^3+3x-6}{x^4+3x^2}dx.$

Solution

Integrand is not a proper rational function so one has to perform the polynomial long division in order to represent integrand as a sum of some polynomial and the proper rational function:

$$
\begin{array}{r|l}
-\;\; 2x^5+6x^3+3x-6 & x^4+3x^2 \\ \cline{2-2}
2x^5+6x^3 \qquad\quad & 2x \\ \cline{1-1}
3x-6 &
\end{array}
$$

then

$$\int \frac{2x^5+6x^3+3x-6}{x^4+3x^2}dx = \int\left(2x+\frac{3x-6}{x^4+3x^2}\right)dx = x^2+\int\frac{3x-6}{x^4+3x^2}dx = x^2+I_1.$$

The remaining integral contains the proper rational function, so we have to factorize denominator and represent integrand as a sum of partial fractions:

$$\frac{3x-6}{x^4+3x^2}=\frac{3x-6}{x^2(x^2+3)}=\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{Mx+N}{x^2+3}=\left\{\begin{array}{l}\text{bring to a common}\\ \text{denominator}\end{array}\right\}=$$

$$=\frac{A_1x(x^2+3)+A_2(x^2+3)+(Mx+N)x^2}{x^2(x^2+3)}.$$

Equate the numerators of the resulting and the original fractions:

$$A_1(x^3+3x)+A_2(x^2+3)+(Mx+N)x^2=3x-6;$$

$$x^3\left(A_1+M\right)+x^2\left(A_2+N\right)+3A_1x+3A_2=3x-6.$$

Equate the coefficients at the same degrees x:

$$\left.\begin{array}{l}x^3:\;\; A_1+M=0,\\ x^2:\;\; A_2+N=0,\\ x^1:\;\; 3A_1=3,\\ x^0:\;\; 3A_2=-6.\end{array}\right\}$$

System solution: $A_1=1,\ A_2=-2,\ M=-1,\ N=2.$ So

$$I_1=\int\frac{3x-6}{x^4+3x^2}dx=\int\left(\frac{1}{x}+\frac{-2}{x^2}+\frac{-x+2}{x^2+3}\right)dx=\int\frac{dx}{x}-2\int\frac{dx}{x^2}+\int\frac{-x+2}{x^2+3}dx=$$

$$=\int\frac{dx}{x}-2\int x^{-2}dx-\int\frac{xdx}{x^2+3}+2\int\frac{dx}{x^2+3}=$$

$$=\ln|x|+\frac{1}{x}-\frac{1}{2}\ln\left|x^2+3\right|+\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}}+C.$$

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Here we used formulas 3, 2 and 10 from the list of integrals to find the first, the second and the last integral. To evaluate the third integral we did substitution $x^2+3=t$, $2xdx=dt$:

$$\int\frac{xdx}{x^2+3}=\frac{1}{2}\int\frac{dt}{t}=\frac{1}{2}\ln\left|x^2+3\right|.$$

Finally get:

$$\int\frac{2x^5+6x^3+3x-6}{x^4+3x^2}dx=x^2+I_1=$$

$$=x^2+\ln|x|+\frac{2}{x}-\frac{1}{2}\ln\left|x^2+3\right|+\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}}+C.$$

Exercises

In the exercises 237–242 evaluate the following integrals:

237. $\int\frac{2x^2+1}{(x^2+8x-9)(x-2)}dx.$

238. $\int\frac{4-3x}{(x+2)^2(x-1)}dx.$

239. $\int\frac{5x+3}{(x^2+2)(x+2)}dx.$

240. $\int\frac{3x^2+1}{x^3+x}dx.$

241. $\int\frac{7x-4}{x^3+6x^2+12x+8}dx.$

242. $\int\frac{x^5}{x^3+2x^2+2x}dx.$

4.7. Integration of Trigonometric Functions

Type 1. Integrals of the form

$$I=\int\sin^m x\cos^n xdx$$

are found by applying different techniques depending on the values of *m* and *n*:

1. At least one of the numbers m or n is odd.

Let $n=2l+1$. Then

$$I=\int\sin^m x\cos^{2l+1}xdx=\int\sin^m x\cos^{2l}x\cos xdx=$$

$$=\int\sin^m x\left(\cos^2 x\right)^l\cos xdx=\left\{\begin{array}{l}t=\sin x,\ dt=\cos xdx\\ \sin^2 x+\cos^2 x=1\Rightarrow\\ \cos^2 x=1-\sin^2 x=1-t^2\end{array}\right\}=\int t^m\left(1-t^2\right)^l dt.$$

The same is done in the case when m is odd.

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Example 4.9. Evaluate the integral $\int \sin^3 x \cos^2 x dx$.

Solution

$$\int \sin^3 x \cos^2 x dx = \int \sin^2 x \cos^2 x \sin x dx = \left\{ \begin{matrix} t = \cos x, \ \ dt = -\sin x dx \\ \sin^2 x = 1 - \cos^2 x = 1 - t^2 \end{matrix} \right\} =$$

$$= -\int (1 - t^2) t^2 dt = -\int (t^2 - t^4) dt = -\left(\frac{t^3}{3} - \frac{t^5}{5} \right) + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.$$

2. Both numbers m and n are even. Then one have to use formulas:

$$\sin^2 x = \frac{1 - \cos 2x}{2}; \qquad (4.2)$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}; \qquad (4.3)$$

$$\sin x \cos x = \frac{1}{2} \sin 2x. \qquad (4.4)$$

Example 4.10. Evaluate the integral $\int \sin^2 x \cos^2 x dx$.

Solution

Transform integrand by formulas (4.4) and (4.2):

$$\int \sin^2 x \cos^2 x dx = \int (\sin x \cos)^2 dx = \int \left(\frac{1}{2} \sin 2x \right)^2 dx = \frac{1}{4} \int \sin^2 2x dx =$$

$$= \frac{1}{4} \int \frac{1 - \cos 4x}{2} dx = \frac{1}{8} \int (1 - \cos 4x) dx = \frac{1}{8} \left(\int dx - \int \cos 4x dx \right) =$$

$$= \frac{1}{8} \left(x - \frac{1}{4} \sin 4x \right) + C = \frac{x}{8} - \frac{1}{32} \sin 4x + C.$$

Type 2. Integrals of the type

$$\int \mathrm{tg}^m x dx, \quad \int \mathrm{ctg}^m x dx,$$

where m is integer, can be evaluated by means of trigonometric formulas:

$$\mathrm{tg}^2 x = \frac{1}{\cos^2 x} - 1, \ \mathrm{ctg}^2 x = \frac{1}{\sin^2 x} - 1.$$

The degree of tangent or cotangent is consistently reduced by using these formulas.

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Type 3. Rational expressions of trigonometric functions

$$\int R(\cos x, \sin x)dx$$

is reduced to an integral of the rational function of the argument t by substitution

$$\operatorname{tg}\frac{x}{2} = t.$$

Then

$$\sin x = \frac{2t}{1+t^2}, \ \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}.$$

So

$$\int R(\cos x, \sin x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2} = \int R_1(t)dt.$$

Example 4.11 . Evaluate the integral $\int \frac{dx}{8-4\sin x+7\cos x}$.

Solution

$$\int \frac{dx}{8-4\sin x+7\cos x} = \left\{\begin{matrix} \operatorname{tg}\frac{x}{2} = t,\ dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2} \end{matrix}\right\} =$$

$$= \int \frac{\frac{2dt}{1+t^2}}{8-4\cdot\frac{2t}{1+t^2}+7\cdot\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{8(1+t^2)-8t+7(1-t^2)} =$$

$$= 2\int \frac{dt}{t^2-8t+15} = \left\{\begin{matrix} \text{complete the full square} \\ t^2-8t+15 = \left(t^2-2t\cdot 4+4^2\right)-1 = (t-4)^2-1 \end{matrix}\right\} =$$

$$= 2\int \frac{dt}{(t-4)^2-1} = \left\{\begin{matrix} \text{11 from the list} \\ \text{of integrals} \end{matrix}\right\} = 2\cdot\frac{1}{2}\ln\left|\frac{(t-4)-1}{(t-4)+1}\right| + C =$$

$$= \ln\left|\frac{t-5}{t-3}\right| + C = \ln\left|\frac{\operatorname{tg}\frac{x}{2}-5}{\operatorname{tg}\frac{x}{2}-3}\right| + C.$$

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Type 4. Integrals of the form

$$\int \sin ax \cos bx dx,\ \int \cos ax \cos bx dx,\ \int \sin ax \sin bx dx .$$

One have to use formulas:

$$\int \sin ax \cos bx dx = \frac{1}{2}\int \big(\sin(a+b)x + \sin(a-b)x\big)dx;$$

$$\int \cos ax \cos bx dx = \frac{1}{2}\int \big(\cos(a+b)x + \cos(a-b)x\big)dx;$$

$$\int \sin ax \sin bx dx = \frac{1}{2}\int \big(\cos(a-b)x + \cos(a+b)x\big)dx.$$

Example 4.12 . Evaluate the integral $\int \sin 2x \cos 4x dx.$

Solution

$$\int \sin 2x \cos 4x dx = \frac{1}{2}\int \big(\sin 6x - \sin 2x\big)dx = \frac{1}{2}\int \sin 6x dx - \frac{1}{2}\int \sin 2x dx =$$

$$= -\frac{1}{2}\cdot\frac{1}{6}\cos 6x + \frac{1}{2}\cdot\frac{1}{2}\cos 2x + C = -\frac{1}{12}\cos 6x + \frac{1}{4}\cos 2x + C.$$

Exercises

In the exercises 243–252 evaluate the following integrals:

243. $\int \sin^3 2x dx.$

244. $\int \dfrac{\cos^3 x}{\sin x} dx.$

245. $\int \sqrt[3]{\sin^5 x}\cos^5 x dx.$

246. $\int \cos^2 3x dx.$

247. $\int \sin^4 x dx.$

248. $\int \mathrm{tg}\, x dx.$

249. $\int \mathrm{tg}^3 x dx.$

250. $\int \dfrac{dx}{5\sin x + 12\cos x}.$

251. $\int \dfrac{dx}{7\sin x + 4\cos x - 7}.$

252. $\int \cos 3x \cos x dx.$

4.8. Integrals with Radicals

Type 1. Integral

$$I = \int R\left(x, \sqrt[n_1]{x^{m_1}}, \sqrt[n_2]{x^{m_2}}, \ldots, \sqrt[n_k]{x^{m_k}}\right)dx$$

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is reduced to integral of a rational function by substitution

$$x = t^s$$

where s is the smallest common multiple of $n_1, n_2, \ldots, n_k$.

Example 4.13 . Evaluate the integral $\int \frac{\sqrt{x}}{\sqrt{x} - \sqrt[3]{x}} dx.$

Solution

Since $n_1, = 2, n_2 = 3$ then $s = 6$. Then

$$\int \frac{\sqrt{x}}{\sqrt{x} - \sqrt[3]{x}} dx = \left\{ \begin{array}{l} x = t^6 \\ dx = 6t^5 dt \end{array} \right\} = \int \frac{\sqrt{t^6}}{\sqrt{t^6} - \sqrt[3]{t^6}} 6t^5 dt = 6\int \frac{t^3 \cdot t^5}{t^3 - t^2} dt =$$

$$= 6\int \frac{t^8}{t^2(t-1)} dt = 6\int \frac{t^6}{t-1} dt.$$

It is improper rational fraction so we have to divide numerator by denominator (see subsection 4.6). Doing this we get:

$$6\int \frac{t^6}{t-1} dt = 6\int \left(t^5 + t^4 + t^3 + t^2 + t + 1 + \frac{1}{t-1} \right) dx =$$

$$= 6\left(\frac{t^6}{6} + \frac{t^5}{5} + \frac{t^4}{4} + \frac{t^3}{3} + \frac{t^2}{2} + t + \ln|t-1| \right) + C = \left\{ t = \sqrt[6]{x} \right\} =$$

$$= x + \frac{6}{5}\sqrt[6]{x^5} + \frac{3}{2}\sqrt[3]{x^2} + 2\sqrt{x} + 3\sqrt[3]{x} + 6\ln\left|\sqrt[6]{x} - 1\right| + C.$$

Remark. This method can be used for evaluating integrals of the type

$$I = \int R\left(x, \sqrt[n_1]{\frac{ax+b}{cx+d}}, \sqrt[n_{21}]{\frac{ax+b}{cx+d}}, \ldots, \sqrt[n_k]{\frac{ax+b}{cx+d}} \right) dx.$$

In this case we should use the substitution

$$\frac{ax+b}{cx+d} = t^s$$

where s is the smallest common multiple of $n_1, n_2, \ldots, n_k$.

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Type 2. Integrals involving radicals $\sqrt{a^2-x^2}$, $\sqrt{x^2 \pm a^2}$.

Integrals of this type can be evaluated using trigonometric substitutions:

Integral	Substitution
$\int R(x, \sqrt{a^2-x^2})dx$	$x = a\sin t$, $dx = a\cos t dt$, $\sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2 x)} = a\cos t$
$\int R(x, \sqrt{x^2+a^2})dx$	$x = a \text{ tg } t$, $dx = \frac{a}{\cos^2 t}dt$, $\sqrt{x^2+a^2} = \sqrt{a^2(1+\text{tg}^2 t)} = \frac{a}{\cos t}$
$\int R(x, \sqrt{x^2-a^2})dx$	$x = \frac{a}{\cos t}$, $dx = \frac{a\sin t}{\cos^2 t}dt$, $\sqrt{x^2-a^2} = \sqrt{a^2\left(\frac{1}{\cos^2 t}-1\right)} = \frac{a\sin t}{\cos t}$

The integrals take the form:

$$\int R_1(\sin t, \cos t)dt..$$

Evaluation of this type of integrals is discussed in subsection 4.7.

Exercises

In the exercises 253–260 evaluate the following integrals:

253. $\int \frac{dx}{\sqrt[3]{4x-3}+1}$.

254. $\int \frac{\sqrt[4]{x}+\sqrt{x}}{\sqrt{x}+1}dx$.

255. $\int \frac{dx}{\sqrt[3]{x-3}-\sqrt{x-3}}$.

256. $\int \frac{2\sqrt[3]{x-1}+1dx}{\sqrt[6]{x-1}+1}$.

257. $\int \frac{\sqrt{3x-1}+2}{4-\sqrt[3]{3x-1}}dx$.

258. $\int \frac{x+1}{\sqrt{x}+2\sqrt[3]{x}}dx$.

259. $\int \frac{\sqrt{(4-x^2)^3}}{x^6}dx$.

260. $\int \frac{dx}{x^2\sqrt{x^2-1}}$.

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SECTION 5. DEFINITE INTEGRAL

5.1. Basic Concepts

Given a function $y = f(x)$ that is continuous on the interval $[a;b]$. Let's divide up $[a;b]$ into n arbitrary intervals by points:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

The length of each interval is denoted as $\Delta x_i = x_{i+1} - x_i$.

Select the point ξ_i, $x_{i-1} < \xi_i < x_i$ within each partial interval (see Fig. 5.1.) and find the values of the function $y = f(x)$ at points ξ_i.

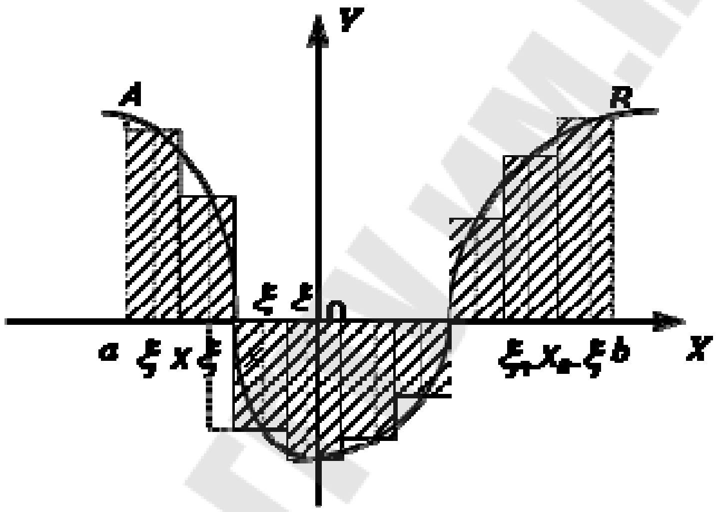

Fig. 5.1

The sum

$$S_n = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \cdots + f(\xi_n)\Delta x_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

is called a Riemann Sum.

Definition. Let $n \to \infty$ and Δx_i tends to 0 for $i = 1, 2, \ldots, n$. If the limit of the Riemann Sum exists and does not depend on a choice of the points x_i and ξ_i, then it is called a definite integral of the function $f(x)$ over the interval $[a;b]$:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

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Numbers a and b that are at the bottom and at the top of the integral sign are called the *lower* and the *upper limits* of the integral respectively.

Geometrically, the sum S_n is the algebraic sum of the areas of rectangles at the bases of which lie partial segments of Δx_i, and the heights are $f(\xi_i)$.

Properties of definite integral:

1. $\int_a^a f(x)dx.$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

4. $\int_a^b Af(x)dx = A\int_a^b f(x)dx.$

5. $\int_a^b \big(f(x) \pm g(x)\big)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$

6. $\int_a^b Cdx = C(b-a)$ for any number C.

7. If $f(x) \geq 0$ for $a < x < b$ then $\int_a^b f(x)dx \geq 0.$

8. If $f(x) \geq g(x)$ for $a < x < b$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx.$

9. If $m < f(x) < M$ for $a < x < b$ then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

10. If $f(x)$ is continuous function, $m < f(x) < M$ for $a < x < b$ then

$$\int_a^b f(x)dx = f(c)(b-a)$$

for some number $c \in (a; b)$.

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11. If $f(x)$ is even, then $f(x)=f(-x)$ and $\int\limits_{-a}^{a} f(x)dx = 2\int\limits_{0}^{a} f(x)dx.$

12. If $f(x)$ is even, then $f(x)=-f(-x)$ and $\int\limits_{-a}^{a} f(x)dx = 0.$

Newton–Leibniz axiom

Suppose $f(x)$ is a continuous function on $[a;b]$ and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then

$$\int\limits_{a}^{b} f(x)dx = F(x)\Big|_a^b = F(b)-F(a). \qquad (5.1)$$

Example 5.1. Calculate the integral $\int\limits_{1}^{2}\left(x^2+3\right)dx.$

Solution

Let's find anti-derivative and use Newton-Leibniz axiom:

$$\int\limits_{1}^{2}\left(x^2+3\right)dx = \int\limits_{1}^{2} x^2dx + 3\int\limits_{1}^{2} dx = \frac{x^3}{3}\Bigg|_1^2 + 3x\Big|_1^2 =$$

$$= \left\{\begin{matrix} \text{substitute } b=2 \text{ and} \\ a=1 \text{ instead of } x \\ \text{by formula (5.1)} \end{matrix}\right\} = \left(\frac{2^3}{3}-\frac{1^3}{3}\right)+\left(3\cdot 2-3\cdot 1\right)=\frac{7}{3}+3=5\frac{1}{3}.$$

When calculating a definite integral, the same techniques are used as when finding an indefinite integral, namely, substitution and the integration by parts formula.

Example 5.2 . Calculate the following integrals:

a) $\int\limits_{0}^{\pi} x\cdot\cos x dx;$ b) $\int\limits_{4}^{9}\frac{dx}{1+\sqrt{x}}.$

Solution

a) To calculate the antiderivative, we apply the method of integration by parts. For a definite integral, the formula for integration by parts takes a form:

$$\int\limits_{a}^{b} udv = uv\Big|_a^b - \int\limits_{a}^{b} vdu.$$

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Then

$$\int_0^{\pi} x\sin 2x dx = \left\{\begin{matrix} u = x, & du = dx \\ dv = \sin 2x dx, & v = -\frac{1}{2}\cos 2x \end{matrix}\right\} =$$

$$= -\frac{1}{2}x\cos 2x\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi}\cos 2x dx = -\frac{1}{2}(\pi\cdot\cos 2\pi - 0\cdot\cos 0) + \frac{1}{4}\sin 2x\Big|_0^{\pi} =$$

$$= -\frac{\pi}{2} + \frac{1}{4}(\sin 2\pi - \sin 0) = -\frac{\pi}{2}.$$

b) Let's change the variable: $t = \sqrt{x}$. Then

$$t^2 = x;$$

$$2tdt = dx.$$

Since the integration limits – 4 and 9 – are given for the variable x, and we are moving to a new variable t, we need to change them according to the formula $t = \sqrt{x}$:

$$t_1 = \sqrt{4} = 2, \quad t_2 = \sqrt{9} = 3.$$

Then

$$\int_4^9 \frac{dx}{1+\sqrt{x}} = \int_2^3 \frac{2tdt}{1+t} = 2\int_2^3 \frac{tdt}{1+t} = 2\int_2^3 \frac{(t+1)-1}{1+t}dt = 2\int_2^3 \frac{t+1}{1+t}dt - 2\int_2^3 \frac{dt}{1+t} =$$

$$= \int_2^3 dt - 2\int_2^3 \frac{dt}{1+t} = t\Big|_2^3 - 2\ln|1+t|\Big|_2^3 = (3-2) - 2(\ln 4 - \ln 3) = 1 - 2\ln\frac{4}{3}.$$

5.2. Geometric Applications of Definite Integrals

Type 1. The area of a region.

The application of definite integral to calculate the area of a flat figure is based on the ***geometric sense*** *of a definite integral*:

If $y = f(x)$ is positive on the interval $[a; b]$ then the definite integral $\int_a^b f(x)dx$ is equal to the *area of a curvilinear trapezoid*, bounded from

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above by the graph of function $y = f(x)$, from below – by a segment of the axis Ox, on the left and right – by straight lines $x = a$ and $x = b$.

If figure is bounded by lines $y = f_1(x)$, $y = f_2(x)$ and two vertical lines $x = a$ and $x = b$, $f_1(x) \le f_2(x)$ on the interval $[a;b]$, the area can be calculated by the following formula:

$$S = \int_a^b \left(f_2(x) - f_1(x)\right)dx. \qquad (5.2)$$

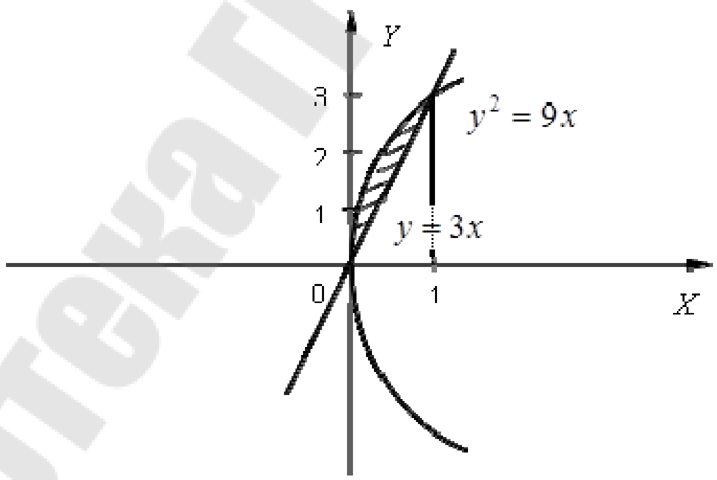
If figure is bounded by the curve given by the parametric equations $x = x(t)$, $y = y(t)$, the area of the curvilinear trapezoid can be calculated by the formula:

$$S = \int_{t_1}^{t_2} y(t)x'(t)dt.$$

Example 5.3. Find the area of the region bounded by the graphs of the functions $y^2 = 9x$, $y = 3x$.

Solution

a) Let's draw a given region:



b) To use the formula (5.2) we need to know limits of integration a and b. This numbers are the abscissas of the intersection points of the given curves:

$$\begin{cases} y^2 = 9x, \\ y = 3x \end{cases} \Rightarrow (3x)^2 = 9x \Rightarrow 9x^2 - 9x = 0;$$

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$$9x(x-1)=0;$$

$$x_1=0,\ x_2=1.$$

c) Taking as $f_1(x)$ function $y=\sqrt{9x}=3\sqrt{x}$ and as $f_2(x)$ function $y=3x$ find the area of the shaded figure:

$$S=\int_0^1\left(3\sqrt{x}-3x\right)dx=3\frac{x^{3/2}}{3/2}\Bigg|_0^1-3\frac{x^2}{2}\Bigg|_0^1=2\sqrt{x^3}\Big|_0^1-\frac{3}{2}x^2\Big|_0^1=2-\frac{3}{2}=\frac{1}{2}.$$

Type 2. The arc length of a curve.

Let the curve be given by the equation $y=f(x)$, where $f(x)$ is a continuously differentiable function, with the abscissas of points A and B equal to $x=a$ and $x=b$, respectively. Then the arc length $\overset{\frown}{AB}$ can be found by the formula:

$$L=\int_a^b\sqrt{1+\left(f'(x)\right)^2}\,dx.$$

If the curve is given by the parametric equations $x=x(t),\ y=y(t)$ then the arc length $\overset{\frown}{AB}$ of the curve is calculated by the formula:

$$L=\int_{t_1}^{t_2}\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt.$$

Exercises

In the exercises 261–263 calculate the following definite integrals:

261. $\int_1^2 x\sqrt{x^2-1}dx.$ 262. $\int_e^{e^2}\frac{dx}{x\ln x}.$ 263. $\int_0^{5\pi}x\cdot\sin\frac{x}{10}dx.$

In the exercises 264–269 find the area bounded by the following curves:

264. $y=x^2-4,\quad y=3x.$

265. $y=\sqrt{x},\ x+y=2,\ x=0.$

266. $y=3x-x^2,\ y=-x.$

267. $xy=2, x=1,\ x=2,\ y=4.$

268. $y=x^3,\quad y=\sqrt{x}.$

269. $\begin{cases}x=3\cos^3 t,\\ y=3\sin^3 t.\end{cases}$

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270. Find the arc length of a curve $y=\frac{1}{3}\sqrt{(2x-1)^3}$ between points $x=2$ and $x=8$.

271. Find the arc length of a curve $\begin{cases} x=4(t-\sin t) \\ y=4(1-\cos t) \end{cases}$, $0\le t\le 2\pi$.

5.3. Improper Integrals

Introducing the concept of a definite integral, we assumed that two conditions are satisfied: (i) the interval of integration is finite and (ii) the function is continuous on it. If at least one of these conditions is violated, then the integral is called *improper*.

Type 1. Infinite interval.

In this type of integral one or both of the limits of integration are infinity.

If $y=f(x)$ exists for every $x\ge a$ then

$$\int_a^{+\infty} f(x)dx=\lim_{t\to\infty}\int_a^t f(x)dx.$$

In a similar way one can define improper integrals $\int_{-\infty}^{b} f(x)dx$ and $\int_{-\infty}^{+\infty} f(x)dx.$

We call improper integral *convergent* if the associated limit exists and is a finite number (i. e. it's not plus or minus infinity) and *divergent* if the associated limit either doesn't exist or is (plus or minus) infinity.

Example 5.4. Determine if the integral $\int_1^{+\infty}\frac{dx}{\sqrt{x}}$ is convergent or divergent.

Solution

$$\int_1^{+\infty}\frac{dx}{\sqrt{x}}=\lim_{t\to+\infty}\int_1^t x^{-\frac{1}{2}}dx=\lim_{t\to+\infty}\left.\frac{x^{\frac{1}{2}}}{1/2}\right|_1^t=\lim_{t\to+\infty}\left(2\sqrt{t}-2\sqrt{1}\right)=+\infty,$$

hence integral is divergent.

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Type 2. Discontinuous integrand.

If $y = f(x)$ is continuous on the interval $[a;b)$ and not continuous at a point $x = b$, then

$$\int_a^b f(x)dx = \lim_{t \to b-0} \int_a^t f(x)dx.$$

We do need to use a left-hand limit here since the interval of integration is entirely on the left side of the upper limit.

In a similar way one can define improper integral if $y = f(x)$ is continuous on the interval $(a;b]$ and not continuous at a point $x = a$:

$$\int_a^b f(x)dx = \lim_{t \to a+0} \int_t^b f(x)dx.$$

Example 5.5. Determine if the integral $\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent or divergent.

Solution

The integrand is not continuous at the point $x = 0$, so

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t \to +0} \int_t^1 x^{-\frac{1}{2}} dx = \lim_{t \to +0} \left. \frac{x^{\frac{1}{2}}}{1/2} \right|_t^1 = \lim_{t \to +0} \left(2\sqrt{1} - 2\sqrt{t} \right) = 2,$$

hence integral is convergent.

Exercises

In the exercises 272–277 determine if the following integrals are convergent or divergent:

272. $\int_e^{+\infty} \frac{dx}{x \ln^2 x}$.

273. $\int_e^{+\infty} \frac{dx}{x \ln x}$.

274. $\int_2^5 \frac{dx}{(x-2)^3}$.

275. $\int_3^{+\infty} \frac{dx}{x^2 - 4}$.

276. $\int_2^5 \frac{dx}{(x-2)^3}$.

277. $\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$.

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SECTION 6. ORDINARY DIFFERENCIAL EQUATIONS

6.1. Basic Concepts

Definition. A *differential equation* is any equation which contains derivatives, either ordinary derivatives or partial derivatives. A differential equation is called an *ordinary differential equation*, if it contains ordinary derivatives. Ordinary differential equation (ODE) has a single independent variable.

The *order* of a differential equation is the largest derivative present in the differential equation.

A *solution* of a differential equation is any function $y(x)$ which satisfies the given differential equation.

Initial conditions (often abbreviated i. c.) are values of the solution and/or its derivative(s) at specific point. For an n-th order equation i. c. have a form

$$\begin{cases} y(x_0) = y_0, \\ y'(x_0) = y_1, \\ \dots\dots\dots\dots \\ y^{(n-1)}(x_0) = y_{n-1}. \end{cases}$$

The *general solution* to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

The general solution for an n-th order equation is usually written as

$$y = \varphi(x, C_1, \dots, C_n), \text{ or } \Phi(x, y, C_1, \dots, C_n) = 0,$$

where $C_1, \dots, C_n$ are arbitrary constants.

The *actual solution* to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

An *Initial Value Problem* (or IVP) is a differential equation along with an appropriate number of initial conditions.

In the case of n-th order DE IVP can be written as:

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$$\begin{cases} F(x, y, y', y'', \ldots, y^{(n)}) = 0, \\ y(x_0) = y_0, \\ y'(x_0) = y_1, \\ \ldots\ldots\ldots\ldots \\ y^{(n-1)}(x_0) = y_{n-1}. \end{cases}$$

The solution of IVP is the actual solution.

6.2. First Order Differential Equations

The first order differential equation is a relation of the form

$$F(x, y, y') = 0,$$

or in deferential form:

$$M_1(x) \cdot N_1(y)dx + M_2(x) \cdot N_2(y)dy = 0.$$

One can find a general solution of differential equation in a form $y = \varphi(x, C)$ (or $\Phi(x, y, C) = 0$) where C is arbitrary constant. In the case of first order differential equation IVP can be written as:

$$\begin{cases} F(x, y, y') = 0; \\ y(x_0) = y_0. \end{cases}$$

Consider some types of first-order differential equations.

6.2.1. Separable Differential Equations

A *separable differential equation* is any differential equation that we can write in the following form

$$y' = f(x) \cdot g(y), \tag{6.1}$$

or

$$M_1(x) \cdot N_1(y)dx + M_2(x) \cdot N_2(y)dy = 0.$$

To solve the equation (6.1) you need to do the following steps:

Step 1: write $y' = \dfrac{dy}{dx}$,

$$\frac{dy}{dx} = f(x) \cdot g(y).$$

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Step 2: multiply both parts by dx:

$$dy = f(x) \cdot g(y)dx.$$

Step 3: divide both parts by $g(y)$:

$$\frac{dy}{g(y)} = f(x)dx. \qquad (6.2)$$

Equation (6.2) is a differential equation with separated variables.

Step 4: integrate one side with respect to y and the other side with respect to x:

$$\int \frac{dy}{g(y)} = \int f(x)dx;$$

$$\int \frac{dy}{g(y)} = G(y) + C_1, \ \int f(x)dx = F(x) + C_2.$$

Step 5: substitute the results of integration and simplify:

$$G(y) + C_1 = F(x) + C_2,$$

$$G(y) - F(x) = C_2 - C_1.$$

Implicit solution for separable differential equation is:

$$G(y) - F(x) = C, \text{ where } C = C_2 - C_1.$$

Example 6.1. Solve the following IVP: $y' = (2y+1)\cos 4x$, $y(\pi) = 0$.

Solution

Step 1: $\frac{dy}{dx} = (2y+1)\cos 4x.$

Step 2: multiply both parts by dx:

$$dy = (2y+1)\cos 4x\, dx.$$

Step 3: divide both parts by $(2y+1)$:

$$\frac{dy}{2y+1} = \cos 4x\, dx.$$

Step 4: integrate both sides:

$$\int \frac{dy}{2y+1} = \frac{1}{2}\ln|2y+1| + C_1;$$

$$\int \cos 4x dx = \frac{1}{4}\sin 4x + C_2.$$

Step 5: substitute the results of integration and simplify:

$$\frac{1}{2}\ln|2y+1| + C_1 = \frac{1}{4}\sin 4x + C_2.$$

Thus, the implicit solution for given differential equation is:

$$\frac{1}{2}\ln|2y+1| - \frac{1}{4}\sin 4x = C.$$

The solution of IVP is the actual solution when $y(\pi) = 0$:

$$\frac{1}{2}\ln|2\cdot 0+1| - \frac{1}{4}\sin 4\pi = C,\ \ C = 0.$$

The solution of IVP is

$$\frac{1}{2}\ln|2y+1| - \frac{1}{4}\sin 4x = 0 \text{ or } 2\ln|2y+1| - \sin 4x = 0.$$

Exercises

In the exercises 278–283 find a general solutions of ODE:

278. $y' = (2x-1)\text{ctg}y.$

279. $y' = e^{x^2} x\left(1+y^2\right).$

280. $\text{ctg}\, x\cos^2 y + \sin^2 x\text{tg}\, y = 0.$

281. $xydx + \left(x^2+1\right)dy = 0.$

282. $y' = y\ln y.$

283. $y' = 2^{x-y}.$

In the exercises 284–287 solve the following IVP:

284. $(x+4)dy - xydx = 0,\ \ y(-3) = 1.$

285. $y' = 2xy + x,\ \ y(0) = 0.$

286. $y^2\ln x dx - (y-1)x dy = 0,$
$y(1) = 1.$

287. $y'\sqrt{1-x^2} - \cos^2 y = 0,$
$y(0) = 0.$

6.2.2. Homogeneous Differential Equations

A function $f(x, y)$ is called *homogeneous* of degree n if, for any λ, the following condition is satisfied:

$$f(\lambda x, \lambda y) = \lambda^n f(x, y).$$

A first order differential equation

$$y' = f(x, y) \qquad (6.3)$$

is *homogeneous* when $f(x, y)$ is homogeneous function of zero degree, that is

$$f(\lambda x, \lambda y) = f(x, y).$$

To solve the homogeneous DE is necessary to create a new variable

$$u(x) = \frac{y(x)}{x}.$$

After substitution

$$y = ux,$$

$$y' = (ux)' = u'x + ux' = u'x + u$$

into the equation (6.3) we obtain separable differential equation (see 6.2.1).

Example 6.2. Find the general solution of the DE $x^2 y' = y(x + y)$.

Solution

Let’s express y':

$$y' = \frac{y(x+y)}{x^2}.$$

Let's check whether the function $f(x, y) = \dfrac{y(x+y)}{x^2}$ is homogeneous:

$$f(\lambda x, \lambda y) = \frac{\lambda y(\lambda x + \lambda y)}{(\lambda x)^2} = \frac{\lambda^2 y(x+y)}{\lambda^2 x^2} = \frac{y(x+y)}{x^2} = f(x, y),$$

hence, given equation is the homogeneous differential equation.

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Step 1: we create a new variable $u(x) = \dfrac{y(x)}{x}$,

$$y = ux;$$

$$y' = u'x + u.$$

Step 2: substitute into a given equation:

$$u'x + u = \frac{ux(x + ux)}{x^2};$$

$$u'x + u = \frac{ux^2(1 + u)}{x^2};$$

$$u'x + u = \frac{u(1 + u)}{1} \Rightarrow u'x + u = u + u^2;$$

$$u'x = u^2.$$

Step 3: solve the resulting separable equation:

$$u' = \frac{u^2}{x} \Rightarrow \frac{du}{dx} = \frac{u^2}{x} \Rightarrow \frac{du}{u^2} = \frac{dx}{x};$$

$$\int \frac{du}{u^2} = \int u^{-2} du = \frac{u^{-1}}{-1} = -\frac{1}{u}, \int \frac{dx}{x} = \ln|x|.$$

Thus

$$-\frac{1}{u} = \ln|x| + \ln C \text{ or } -\frac{1}{u} = \ln Cx.$$

Note that here we have used the constant $\ln C$ instead of C. It is convenient to do this when one of the integrals (or both) contains the logarithm.

Step 4: return to the initial variable $y = ux$:

$$-\frac{1}{y/x} = \ln Cx.$$

So the general solution is $y = -\dfrac{x}{\ln Cx}$.

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Exercises

In the exercises 288–293 solve the following DE:

288. $y' = \frac{x}{y} + \frac{y}{x}$.

289. $y' = 4 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$.

290. $(x - y)dx + xdy = 0$.

291. $(x^2 + y^2)dy - 2xydx = 0$.

292. $xy' = x\sin\frac{y}{x} + y$.

293. $xy' - y = x\text{tg}\left(\frac{y}{x}\right)$.

6.2.3. First Order Linear Differential Equations

A first order differential equation is linear when it can be made to look like this:

$$y' + p(x)y = q(x), \qquad (6.4)$$

where $p(x)$ and $q(x)$ are both continuous functions.

To solve it there is a special method:

Step 1: find a solution in a form $y = u(x) \cdot v(x)$ where u and v are functions of x. Then

$$y' = (uv)' = u' \cdot v + u \cdot v'.$$

Step 2: substitute y and y' into a given equation (6.4):

$$u'v + uv' + p(x)uv = q(x).$$

Step 3: factor the parts involving u:

$$u'v + u(v' + p(x)v) = q(x).$$

Step 4: put expression in brackets equal zero. So, we get a system:

$$\begin{cases} v' + p(x)v = 0, \\ u'v = q(x). \end{cases}$$

Step 5: the first equation is separable DE. Solve the first, and after the second equation, find u and v.

Step 6: finally, substitute u and v into $y = u \cdot v$ to get solution.

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Example 6.3. Find the general solution of the DE $y' + \frac{3y}{x} = x^2$.

Solution

It is Linear Differential Equation in a form (6.4).
Find a solution as $y = uv$, then $y' = u'v + uv'$.
Substitute y and y' into equation:

$$u'v + uv' + \frac{3}{x}uv = x^2;$$

$$u'v + u\left(v' + \frac{3v}{x}\right) = x^2.$$

Put expression in brackets equal zero and obtain a system:

$$\begin{cases} v' + \frac{3v}{x} = 0, \\ u'v = x^2. \end{cases}$$

Solve separable differential equation $v' + \frac{3v}{x} = 0$:

$$\frac{dv}{dx} = -\frac{3v}{x} \Rightarrow \frac{dv}{v} = -\frac{3dx}{x} \Rightarrow \int \frac{dv}{v} = -3\int \frac{dx}{x}.$$

Since

$$\ln v = -3\ln x = \ln x^{-3}.$$

Then $v = x^{-3} = \frac{1}{x^3}$.

Substitute $v = \frac{1}{x^3}$ into the second equation of a system:

$$\frac{u'}{x^3} = x^2 \Rightarrow u' = x^5.$$

Integrating the last expression, we get

$$u = \int x^5 dx = \frac{x^6}{6} + C.$$

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Finally, substitute u and v into $y = u \cdot v$ to get general solution:

$$y = u \cdot v = \left(\frac{x^6}{6} + C \right) \frac{1}{x^3} = \frac{x^3}{6} + \frac{C}{x^3}.$$

Remark. This method can be used to solve the DE of the form

$$y' + p(x)y = q(x)y^{\alpha},$$

where α is any rational number except 0 and 1 (Bernoulli equation).

Exercises

In the exercises 294–299 solve the following DE:

294. $y' - \frac{2y}{x} = x^2 + 1.$

295. $xy' - 2y = 2x^4.$

296. $y' - y\,\mathrm{tg}x = -y^2 \cos x.$

297. $y' + 2xy = xe^{-x^2}.$

298. $\begin{cases} y' + \frac{y}{x} = x^5 - 2, \\ y(1) = \frac{1}{7}. \end{cases}$.

299. $\begin{cases} xy' - 2y = x^3 \cos x, \\ y(\pi) = 1. \end{cases}$

6.3. Second Order Linear Differential Equations

The second order linear differential equation is

$$y'' + a_1 y' + a_2 y = f(x), \tag{6.5}$$

where $a_1(x), a_2(x), f(x)$ are known functions.

If $f(x) = 0,$ then we get a *homogeneous linear equation*

$$y'' + a_1 y' + a_2 y = 0. \tag{6.6}$$

Principle of Superposition . If y_1 and y_2 are any two solutions of the homogeneous equation (6.6) then any function of the form

$$y = C_1 y_1 + C_2 y_2$$

is also a solution of the equation, for any pair of constants C_1 and C_2.

Given two non-zero functions $f(x)$ and $g(x)$ are called *linearly independent* if the only two constants for which equation

$$C_1 f(x) + C_2 g(x) = 0$$

is true are $C_1 = 0$ and $C_2 = 0.$

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Theorem 6.1. If y_1 and y_2 are two linearly independent solutions of the homogeneous equation (6.6) then function

$$y = C_1 y_1 + C_2 y_2$$

is a general solution of the equation (6.6).

Theorem 6.2. The general solution of linear inhomogeneous equation (6.5) can be written as

$$y = y_0(x) + Y(x),$$

where $y_0(x)$ is the general solution to the corresponding homogeneous equation, also called the complementary solution, $Y(x)$ is any particular solution for given equation.

6.3.1. Second Order Linear Homogeneous Differential Equations with Constant Coefficients

Consider homogeneous linear differential equation

$$y'' + py' + qy = 0 \qquad (6.7)$$

where p and q are known numbers.

Let $y = e^{kx}$ be a solution of equation, for some as-yet-unknown constant k. Then $y' = ke^{kx}$, $y'' = k^2 e^{kx}$.

Substitute y, y' and y'' into equation (6.7):

$$k^2 e^{kx} + pke^{kx} + qe^{kx} = 0;$$

$$k^2 + pk + q = 0. \qquad (6.8)$$

It is the *characteristic equation* of (6.7).

Solve characteristic equation:

$$D = p^2 - 4q,\ x_{1,2} = \frac{-p \pm \sqrt{D}}{2a}.$$

Depending on the discriminant of the characteristic equation, we obtain the general solution of equation (6.7) in one of three following forms:

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Discriminant	***eneral solution***
$D>0$, $k_1 \neq k_2$ are two distinct real roots	$y = C_1 e^{k_1 x} + C_2 e^{k_2 x}$
$D=0$, $k_1 = k_2 = k$ is one repeated real root	$y = e^{kx}\left(C_1 + C_2 \cdot x\right)$
$D=0$, $k_{1,2} = \alpha \pm \beta i$ are two complex conjugate roots	$y = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right)$

Example 6.4 . Find the general solution of the following equation:
a) $y'' - 3y' = 0$; b) $y'' + 6y' + 9y = 0$; c) $y'' + 6y' + 13y = 0$.

Solution

a) Solve the characteristic equation:

$$k^2 - 3k = 0;$$

$$k(k-3) = 0.$$

So $k_1 = 0, k_2 = 3$ are two distinct real roots. The general solution is

$$y = C_1 e^{0 \cdot x} + C_2 e^{3x} = C_1 + C_2 e^{3x}.$$

b) Characteristic equation is

$$k^2 + 6k + 9 = 0;$$

$$D = 6^2 - 4 \cdot 1 \cdot 9 = 0 \Rightarrow k_{1,2} = \frac{-6 \pm 0}{2 \cdot 1} = -3.$$

We get one repeated real root, so the general solution is

$$y = e^{-3x}(C_1 + C_2 x).$$

c) Characteristic equation is

$$k^2 + 6k + 13 = 0;$$

$$D = 6^2 - 4 \cdot 1 \cdot 13 = -16;$$

$$k_{1,2} = \frac{-6 \pm \sqrt{-16}}{2 \cdot 1} = \frac{-6 \pm 4i}{2} = \frac{-6}{2} \pm \frac{4i}{2} = -3 \pm 2i.$$

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We get two complex conjugate roots, $\alpha=-3,\ \beta=2$ so the general solution is

$$y_0=e^{-3x}(C_1\cos 2x+C_2\sin 2x).$$

Example 6.5 . Solve the following IVP:

$$y''+49y=0,\ y\left(\frac{\pi}{2}\right)=2,\ y'\left(\frac{\pi}{2}\right)=7\,.$$

Solution

The characteristic equation is

$$k^2+49=0,$$
$$k^2=-49\Rightarrow k=\pm\sqrt{-49}\Rightarrow k_{1,2}=\pm 7i.$$

We get two complex conjugate roots, $\alpha=0,\ \beta=7,$ so the general solution is

$$y=e^{0x}(C_1\cos 7x+C_2\sin 7x)=C_1\cos 7x+C_2\sin 7x.$$

Let's find actual solution using initial conditions. Find the derivative:

$$y'=-7C_1\sin 7x+7C_2\cos 7x.$$

Plugging in the initial conditions gives the following system:

$$\begin{cases}C_1\cos\frac{7\pi}{2}+C_2\sin\frac{7\pi}{2}=2,\\ -7C_1\sin\frac{7\pi}{2}+7C_2\cos\frac{7\pi}{2}=7.\end{cases}\Rightarrow\begin{cases}C_1\cdot 0-C_2\cdot 1=2,\\ -7C_1\cdot(-1)+7C_2\cdot 0=7.\end{cases}$$

The solution of a system is $C_2=-2, C_1=1.$

The actual solution of differential equation is

$$y=\cos 7x-2\sin 7x.$$

Exercises

In the exercises 300–307 solve the following homogeneous DE:

300. $y''+y'-6y=0.$

301. $y''+4y'=0.$

302. $y''-8y'+16y=0.$

303. $y''-16y=0.$

304. $y''+16y=0.$

305. $y''+4y'+5y=0.$

306. $y''+2y'+5y=0.$

307. $y''+2y'+y=0.$

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6.3.2. Second Order Linear Nonhomogeneous Differential Equations with Constant Coefficients

In some special cases the particular solution $Y(x)$ can be found by *Undetermined Coefficients Method.*

I. Let DE has a form

$$y'' + py' + qy = P_n(x)e^{ax},$$

where $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, and let $k_{1,2}$ are the roots of characteristic equation $k^2 + pk + q = 0$. Then

– if $a \neq k_{1,2}$ then $Y(x)$ has to be taken in a form

$$Y(x) = e^{ax}\left(Ax^n + Bx^{n-1} + \cdots\right);$$

– if $a = k_1$, $a \neq k_2$ then $Y(x)$ has to be taken in a form

$$Y(x) = xe^{ax}\left(Ax^n + Bx^{n-1} + \cdots\right);$$

– if $a = k_1 = k_2 = k$ then $Y(x)$ has to be taken in a form

$$Y(x) = x^2 e^{ax}\left(Ax^n + Bx^{n-1} + \cdots\right),$$

where $A, B, \ldots$ are undefined coefficients.

One have to calculate $Y'(x)$ and $Y''(x)$, plug $Y(x)$, $Y'(x)$ and $Y''(x)$ into given equation and collect the like terms. We will need to choose numbers $A, B, \ldots$ so that the coefficients of the exponentials on either side of the equal sign are the same.

Example 6.6. Find the general solution to $y'' - 4y' - 12y = 3e^{5x}$.

Solution

a) Firstly one should solve corresponding homogeneous equation

$$y'' - 4y' - 12y = 0.$$

The characteristic equation is

$$k^2 - 4k - 12 = 0;$$

$$D = (-4)^2 - 4\cdot(-12) = 64 > 0;$$

$$k_{1,2} = \frac{4 \pm 8}{2},\ k_1 = 6,\ k_2 = -2.$$

So the complementary solution is

$$y_0 = C_1 e^{6x} + C_2 e^{-2x}.$$

b) Find particular solution $Y(x)$. The right side of equation

$$f(x) = 3e^{5x}$$

has a form $f(x) = P_n(x)e^{ax}$ where $n = 0$ and $a = 5 \neq k_{1,2}$. Thus the particular solution $Y(x)$ has to be taken in a form

$$Y(x) = Ae^{5x}.$$

Find $Y'(x)$ and $Y''(x)$, and plug $Y(x)$, $Y'(x)$ and $Y''(x)$ into a given equation:

$$Y'(x) = 5Ae^{5x},\ Y'(x) = 25Ae^{5x};$$

$$25Ae^{5x} - 4 \cdot 5Ae^{5x} - 12Ae^{5x} = 3e^{5x};$$

$$-7Ae^{5x} = 3e^{5x} \Rightarrow -7A = 3 \Rightarrow A = -\frac{7}{3}.$$

Thus $Y(x) = -\frac{7}{3}e^{5x}$. By the theorem 6.2 the general solution is

$$y = y_0 + Y(x) = C_1 e^{6x} + C_2 e^{-2x} - \frac{7}{3}e^{5x}.$$

II. Let DE has a form

$$y'' + py' + qy = e^{ax}\left(M_n(x)\cos bx + N_m(x)\sin bx\right),$$

where are $M_n(x)$ and $N_m(x)$ are polynomials of degrees n and m, and let $k_{1,2}$ are the roots of characteristic equation $k^2 + pk + q = 0$. Then

– if $a \pm bi \neq k_{1,2}$ then $Y(x)$ has to be taken in a form

$$Y(x) = e^{ax}\left(A_L(x)\cos bx + B_L(x)\sin bx\right);$$

– if $a \pm bi = k_{1,2}$ then $Y(x)$ has to be taken in a form

$$Y(x) = xe^{ax}\left(A_L(x)\cos bx + B_L(x)\sin bx\right),$$

where $L = \max\{m, n\}$,

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$$A_L(x) = A_1 x^L + A_2 x^{L-1} + \cdots + A_{L+1},$$

$$B_L(x) = B_1 x^L + A_B x^{L-1} + \cdots + B_{L+1}.$$

Example 6.7. Find the general solution to $y'' + 4y' + 13y = 29\sin 2x.$

Solution

a) At first one should solve corresponding homogeneous equation

$$y'' + 4y' + 13y = 0.$$

The characteristic equation is

$$k^2 + 4k + 13 = 0;$$

$$D = 4^2 - 4 \cdot 13 = -36 < 0;$$

$$k_{1,2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i.$$

So the complementary solution is

$$y_0 = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x).$$

b) Find particular solution $Y(x)$. The right side of equation

$$f(x) = 29\sin 2x = e^{0x}\left(0 \cdot \cos 2x + 29\sin 2x\right)$$

has a form $f(x) = e^{ax}\left(M_n(x)\cos bx + N_m(x)\sin bx\right)$ where $m = n = 0,$ $a \pm bi = 0 \pm 2i \neq -2 \pm 3i.$ Thus the particular solution $Y(x)$ has to be taken in a form

$$Y(x) = A\cos 2x + B\sin 2x.$$

Find $Y'(x)$ and $Y''(x)$, and plug $Y(x)$, $Y'(x)$ and $Y''(x)$ into a given equation:

$$Y'(x) = -2A\sin 2x + 2B\cos 2x;$$

$$Y''(x) = -4A\cos 2x - 4B\sin 2x;$$

$$-4A\cos 2x - 4B\sin 2x + 4\left(-2A\sin 2x + 2B\cos 2x\right) +$$

$$+13\left(A\cos 2x + B\sin 2x\right) = 29\sin 2x.$$

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Open brackets and equate the coefficients near the functions $\sin 2x$ and $\cos 2x$:

$$\cos 2x(-4A+8B+13A)+\sin 2x(-4B-8A+13B)=29\sin 2x;$$

$$\cos 2x(9A+8B)+\sin 2x(-8A+9B)=0\cos 2x+29\sin 2x;$$

$$\left.\begin{array}{ll}\cos 2x: & 9A+8B=0,\\ \sin 2x: & -8A+9B=29.\end{array}\right\}$$

Solve a system (we can use Cramer's formulas):

$$\Delta=\begin{vmatrix}9 & 8\\ -8 & 9\end{vmatrix}=81-(-64)=145;$$

$$\Delta_1=\begin{vmatrix}0 & 8\\ 29 & 9\end{vmatrix}=0-8\cdot 29=-232,,\ A=\frac{-232}{145}=-\frac{8}{5}=-1{,}6;$$

$$\Delta_2=\begin{vmatrix}9 & 0\\ -8 & 29\end{vmatrix}=9\cdot 29-0=261,\ B=\frac{261}{145}=\frac{9}{5}=1{,}8.$$

Thus $Y(x)=-1{,}6\cos 2x+1{,}8\sin 2x..$

By the theorem 6.2 the general solution is

$$y=y_0+Y(x)=e^{-2x}(C_1\cos 3x+C_2\sin 3x)-1{,}6\cos 2x+1{,}8\sin 2x.$$

Exercises

In the exercises 308–315 solve the following inhomogeneous DE:

308. $y''-4y'+4y=6e^{2x}$.

309. $y''-9y=10e^{3x}$.

310. $y''-6y'+9y=9x^2-39x+65$.

311. $y''-2y'=x+3$.

312. $y''-2y'+y=(x+1)e^{x}$.

313. $y''+4y=\sin x$.

314. $y''-7y'+6y=-7\sin x-5\cos x$.

315. $y''+4y'+8y=5\sin x$.

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SECTION 7. SERIES

7.1. Numerical Series: Basic Concepts

Consider a numeric sequence $\{a_n\}$. Sums of first n terms:

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$\dots$$
$$s_n = a_1 + a_2 + \ldots + a_n$$

are called *partial sums*.

Partial sums will form a sequence $\{s_n\}$. The limit of the sequence of partial sums s_n

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n + \cdots.$$

is called an *infinite series*. Element a_i is called n*-th term* of series, number i is called the index of summation or just the index.

If the sequence of partial sums $\{s_n\}$ is convergent and its limit is finite then we also call the infinite series $\sum_{i=1}^{\infty} a_i$ convergent. The finite limit of the sequence of partial sums $\{s_n\}$

$$S = \lim_{n\to\infty} s_n$$

is called the value of the infinite series $\sum_{i=1}^{\infty} a_i$. If the sequence of partial sums, $\{s_n\}$ is divergent and its limit is infinite or doesn't exist then we also call the infinite series $\sum_{i=1}^{\infty} a_i$ divergent.

Examples of series:

1. Geometric Series

$$\sum_{i=1}^{\infty} q^n = q + q^2 + q^3 + \cdots.$$

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Geometric series is convergent if $|q|<1$ and $\sum_{i=1}^{\infty} q^n = \frac{q}{1-q}$.

2. Harmonic Series

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\cdots.$$

Harmonic Series is divergent

3. p-Series (Hyperharmonic Series)

$$\sum_{n=1}^{\infty}\frac{1}{n^p} = 1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots.$$

The ***p*** : p-Series converges if $p>1$ and diverges if $p<1$.

7.2. Convergence Tests

I. Necessary Condition of Convergence : if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

The converse is NOT true.

For instant, consider Harmonic Series: $\sum_{n=1}^{\infty}\frac{1}{n}$. $\lim_{n\to\infty}\frac{1}{n}=0$ but Harmonic Series is known to be divergent.

II. Divergence Test: if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 7.1. Determine if the series $\sum_{n=1}^{\infty}\frac{2n+3}{3n+2}$ is convergent or divergent.

Solution

Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n+3}{3n+2} = \frac{2}{3} \neq 0$ then the series is divergent according to Divergence Test.

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III. Comparison Test: suppose that we have two series with positive terms $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. If $a_n \le b_n$ for all $n \ge n_0$, then

– if $\sum_{n=1}^{\infty} b_n$ is convergent then so is $\sum_{n=1}^{\infty} a_n$;

– if $\sum_{n=1}^{\infty} a_n$ is divergent then so is $\sum_{n=1}^{\infty} b_n$.

Example 7.2. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is convergent or divergent.

Solution

Since $\ln n < n$ for all natural numbers n, then $\frac{1}{\ln n} > \frac{1}{n}$. Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then $\sum_{n=1}^{\infty} \frac{1}{\ln n}$ is also divergent according to Comparison Test.

IV. Limit Comparison Test: suppose that we have two series with positive term $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ define $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$. If c is positive and is finite then either both series converge or both series diverge.

Example 7.3. Determine if the series $\sum_{n=1}^{\infty} \frac{n^2+1}{4n^4+2n}$ is convergent or divergent.

Solution

To use Comparison Test we have to choose series $\sum_{n=1}^{\infty} b_n$ to compare. Note that $n^2+1 \sim n^2$, $4n^4+2n \sim 4n^4$ then consider Hyperharmonic series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2};$$

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{n^2+1}{4n^4+2n}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2+1}{4n^4+2n} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^4+n^2}{4n^4+2n} = \frac{1}{4} < \infty.$$

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Hyperharmonic Series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent by p-Test $(p=2>1)$, thus $\sum_{n=1}^{\infty}\frac{n^2+1}{4n^4+2n}$ is also convergent according to the Limit Comparison Test.

V. Ratio Test: suppose we have the series $\sum_{n=1}^{\infty}a_n$. Define $l=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$. Then

– if $l<1$ then the series is *convergent*;
– if $l>1$ then the series is *divergent*;
– If $l=1$ then the series *may be convergent or divergent*.

This test is useful when the expression for n-th term contains factorials.

Example 7.4 . Determine if the series $\sum_{n=1}^{\infty}\frac{3^n}{n!}$ is convergent or divergent.

Solution

The n-th and $n+1$-th terms of the series are written as

$$a_n=\frac{3^n}{n!},\qquad a_{n+1}=\frac{3^{n+1}}{(n+1)!}.$$

Calculate the limit

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}}=\lim_{n\to\infty}\frac{3^{n+1}n!}{3^n(n+1)!}=\begin{cases}3^{n+1}=3\cdot 3^m\\(n+1)!=n!(n+1)\end{cases}=$$

$$=\lim_{n\to\infty}\frac{3^n\cdot 3\cdot n!}{3^n n!(n+1)}=\lim_{n\to\infty}\frac{3}{(n+1)}=0<1.$$

Thus, given series converges according to the Ratio Test.

VI. Root Test: suppose we have the series $\sum_{n=1}^{\infty}a_n$. Define $l=\lim_{n\to\infty}\sqrt[n]{a_n}$. Then

– if $l < 1$ then the series is *convergent*;
– if $l > 1$ then the series is *divergent*;
– If $l = 1$ then the series *may be convergent or divergent*.

This test is used when the index n is both at the base and in the exponent.

Example 7.5. Determine if the series $\sum\limits_{n=1}^{\infty}\left(\frac{2n+4}{3n+5}\right)^{2n}$ is convergent or divergent.

Solution

Calculate the limit

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+4}{3n+5}\right)^{2n}} = \lim_{n\to\infty}\left(\frac{2n+4}{3n+5}\right)^{2} = \left(\frac{2}{3}\right)^{2} = \frac{4}{9} < 1,$$

hence given series is divergent according to the Root Test.

VII. Integral Test: suppose that $f(x)$ is a continuous, positive and decreasing function on the interval $[k;\infty)$ and $f(n) = a_n$ then

– if $\int\limits_{k}^{+\infty} f(x)dx$ is convergent so is $\sum\limits_{n=k}^{\infty} a_n$;

– if $\int\limits_{k}^{+\infty} f(x)dx$ is divergent so is $\sum\limits_{n=k}^{\infty} a_n$.

Example 7.6. Determine if the series $\sum\limits_{n=2}^{\infty}\frac{1}{n\ln n}$ is convergent or divergent.

Solution

In this case the function we'll use is $f(x) = \frac{1}{x\ln x}$. This function is positive and decreasing on the interval $[2;+\infty)$. Therefore we need to determine convergence or divergence of the corresponding integral:

$$\int\limits_{2}^{+\infty} f(x)dx = \int\limits_{2}^{+\infty}\frac{dx}{x\ln x} = \left[\begin{array}{l} u = \ln x;\ du = \frac{dx}{x} \\ u_1 = \ln 2,\ u_2 = \infty \end{array}\right] = \int\limits_{\ln 2}^{+\infty}\frac{du}{u} = \ln u\Big|_{\ln 2}^{\infty} = \infty.$$

The integral is divergent so the series is also divergent by the Integral Test.

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Exercises

In the exercises 316–333 determine if the following series are convergent or divergent:

316. $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$. 317. $\sum_{n=1}^{\infty}\frac{1+n}{\sqrt{n}+n^3}$. 318. $\sum_{n=1}^{\infty}\frac{2n+3}{n^2+5n+3}$.

319. $\sum_{n=1}^{\infty}\frac{n^2}{3n^3+2}$. 320. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^7+2n^4}+n}$. 321. $\sum_{n=1}^{\infty}\frac{2n+3}{n+1}$.

322. $\sum_{n=1}^{\infty}\frac{n^5}{5^n}$. 323. $\sum_{n=1}^{\infty}\frac{n+5}{n!}$. 324. $\sum_{n=1}^{\infty}\frac{(n+1)!}{10^n}$.

325. $\sum_{n=1}^{\infty}\frac{n!\cdot 2^n}{(n+1)!}$. 326. $\sum_{n=0}^{\infty}\left(\frac{n+2}{3n+2}\right)^n$. 327. $\sum_{n=1}^{\infty}\frac{5^n}{3^n n^2}$.

328. $\sum_{n=1}^{\infty}\left(\frac{2n-3}{4n+10}\right)^{n^2}$. 329. $\sum_{n=1}^{\infty}\left(\frac{n^2-3n+2}{2n^2-4n-5}\right)^n$. 330. $\sum_{n=1}^{\infty}\frac{2^n}{2^n+8^n}$.

331. $\sum_{n=1}^{\infty}\left(\frac{3n-2}{3n+1}\right)^n$. 332. $\sum_{n=2}^{\infty}\frac{1}{n\ln^3 n}$. 333. $\sum_{n=1}^{\infty}\sin\frac{1}{n}$.

7.3. Alternating Series

Definition. An alternating series is any series, $\sum_{n=1}^{\infty} b_n$ for which the series terms can be written in one of the following two forms:

$$b_n = (-1)^n a_n \text{ or } b_n = (-1)^{n+1} a_n \ \ (a_n > 0).$$

An alternating series are:

$$\sum_{n=1}^{\infty}(-1)^n a_n = -a_1 + a_2 - a_3 + \cdots \text{ or } \sum_{n=1}^{\infty}(-1)^{n+1} a_n = a_1 - a_2 + a_3 - \cdots.$$

Like any series, an alternating series converges if and only if the associated sequence of partial sums converges.

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Definition. A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely* convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent. If $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent we call the series *conditionally* convergent.

Alternating Series Test (Leibniz Test).

Let we have a series $\sum_{n=1}^{\infty} (-1)^n a_n$ where $a_n > 0$ for all n. Then if

1) $\lim_{n\to\infty} a_n = 0$;

2) $a_n > a_{n+1} > a_{n+2} > \cdots$

the series $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent.

Example 7.7. Determine if the following series are absolute convergent, conditionally convergent or divergent:

a) $\sum_{n=1}^{\infty} (-1)^n \frac{n^6}{6^n}$; b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+5}}$; c) $\sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{5n+2}$.

Solution

a) Let's see if it is an absolutely convergent series:

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n^6}{6^n} \right| = \sum_{n=1}^{\infty} \frac{n^6}{6^n}.$$

Let's use Root Test:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^6}{6^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^6}}{6} = \left\{ \begin{matrix} \text{note that} \\ \lim_{n\to\infty} \sqrt[n]{n} = 1 \end{matrix} \right\} = \frac{1}{6} < 1.$$

Hence, by the Ratio Test this series converges. Thus, given series converges absolutely.

b) Let's see if it is an absolutely convergent series:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n+5}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}} = \sum_{n=6}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=6}^{\infty} \frac{1}{n^{1/2}}.$$

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Hyperharmonic Series with $p=\frac{1}{2}<1$ is divergent by *p*-test, so given series diverges absolutely.

Let's check the conditions of Leibniz Test:

1) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n+5}} = 0;$

2) $a_n = \frac{1}{\sqrt{n+5}},\ a_{n+1} = \frac{1}{\sqrt{n+6}}.$

As $n+5<n+6$ then $a_n = \frac{1}{\sqrt{n+5}} > \frac{1}{\sqrt{n+6}} = a_{n+1}$ for all natural *n*.

Both conditions are fulfilled, so given series is conditionally convergent.

c) Let's find $\lim\limits_{n\to\infty} a_n$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n+1}{5n+2} = \frac{2}{5} \neq 0.$$

So, the Necessary Condition of Convergence is not fulfilled then this series diverges.

Exercises

In the exercises 334–342 determine if the following series are absolute convergent, conditionally convergent or divergent:

334. $\sum\limits_{n=1}^{\infty}(-1)^n\frac{5n+3}{7n+2}$. 335. $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$. 336. $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)}$.

337. $\sum\limits_{n=0}^{\infty}(-1)^n\left(\frac{5n+3}{7n+2}\right)^{2n}$. 338. $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n\sqrt{n}}$. 339. $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[5]{n^3+n}}$.

340. $\sum\limits_{n=1}^{\infty}(-1)^n \operatorname{tg}\left(\frac{1}{n^2}\right)$. 341. $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n+\sqrt{n}}$. 342. $\sum\limits_{n=1}^{\infty}(-1)^n\cos\left(\frac{1}{n}\right)$.

7.4. Functional Series

Suppose $\{u_n(x): n=0,1,2,\ldots\}$ is a sequence of functions defined on an interval *I*. By a functional series we mean the symbol

$$\sum_{n=0}^{\infty}u_n(x) = u_0(x)+u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots\ .$$

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Examples of functional series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots;$$

$$\sum_{n=1}^{\infty} \frac{1}{n^x} = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \cdots.$$

Definition The point x_0 is called the *convergence point* of the functional series $\sum_{n=0}^{\infty} u_n(x)$ if the numerical series

$$\sum_{n=0}^{\infty} u_n(x) = u_0(x_0) + u_1(x_0) + u_2(x_0) + \ldots$$

is convergent. The set of all its points of convergence is called the *domain of convergence* of a functional series.

For example, we know that geometrical series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \tag{7.1}$$

is convergent if $|x| < 1$ and it is divergent if $|x| \geq 1$. Then, the domain of converges is $x \in (-1; 1)$.

Definition. Sum of first n terms of the series is called the n-th partial sum of the functional series. We say that the sequence of the partial sum $\{S_n(x)\}$ *converges pointwise* to the function $\{S(x)\}$ on the interval I if

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \left(u_0(x) + u_1(x) + \cdots + u_n(x)\right).$$

For example the n-th partial sum of the geometric series (7.1) is

$$S_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^n - 1}{x - 1}.$$

The limit of $\{S_n(x)\}$ (in the domain of converges) is

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{x^n - 1}{x - 1} = \frac{1}{1 - x}.$$

Thus, given functional series *converges pointwise to function* $\frac{1}{1-x}$.

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7.5. Power Series

Definition A *power series* in the variable x is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots.$$

Numbers C_0, C_1, C_2 are called *the coefficients of series*.

A power series about a or a general power series is any series that can be written in the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

A general power series always converges at $x = a$. The number a is called *the center of convergence.*

Theorem. For a power series $\sum_{n=0}^{\infty} c_n x^n$ there are three possibilities:

– the power series diverges for all $x \neq 0$;

– the power series converges for all values of x;

– there is a positive number R such that power series converges for all values of x with $|x| < R$ and diverges for all values of x with $|x| > R$. Number R is called *the radius of convergence* of the power series.

If among the coefficients c_n there are no equal to zero, then the radius of convergence can be found using one of the formulas:

$$R = \lim_{n\to\infty} \frac{c_n}{c_{n+1}}; \qquad (7.2)$$

$$R = \lim_{n\to\infty} \frac{1}{\sqrt{c_n}}. \qquad (7.3)$$

The interval of convergence is: $x \in (-R; R)$. One have to investigate the cases $x = \pm R$ separately.

In the case of a power series about a the interval of convergence is $x \in (a-R; a+R)$ and one should investigate the cases $x = a \pm R$ separately.

Example 7.8. Determine the radius of convergence and interval of convergence for the power series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{4^n (n+3)}$.

Solution

The coefficient of series is $c_n = \dfrac{1}{4^n(n+3)}$, $a = 2$.

By formula (7.3)

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{c_n}} = \lim_{n\to\infty} \sqrt[n]{4^n(n+3)} = 4 \lim_{n\to\infty} \sqrt[n]{n+3}.$$

Remained that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, then $\lim\limits_{n\to\infty} \sqrt[n]{n+3} = 1$ to. So, $R = 4 \cdot 1 = 4$.

The interval of convergence is $(a-R; a+R) = (2-4; 2+4) = (-2; 6)$.

Let's investigate series behavior with $x=-2$ and $x=6$.

For $x=6$ we get

$$\sum_{n=0}^{\infty} \frac{(6-2)^n}{4^n(n+3)} = \sum_{n=0}^{\infty} \frac{4^n}{4^n(n+3)} = \sum_{n=0}^{\infty} \frac{1}{n+3}.$$

Let's compare this series with Harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ which is divergent:

$$c = \lim_{n\to\infty} \frac{\frac{1}{n+3}}{\frac{1}{n}} = 1 < \infty, \ c \neq 0.$$

So $\sum\limits_{n=0}^{\infty} \dfrac{1}{n+3}$ is divergent by the Limit Comparison Test.

For $x=-2$ we get

$$\sum_{n=0}^{\infty} \frac{(-2-2)^n}{4^n(n+3)} = \sum_{n=0}^{\infty} \frac{(-4)^n}{4^n(n+3)} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{4^n(n+3)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}.$$

It is alternating series. It diverges absolutely by the Limit Comparison Test. Let's use Leibniz Test:

1) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{n+3} = 0;$

2) $a_n = \dfrac{1}{n+3} > \dfrac{1}{n+4} = a_{n+1}$ for all natural n.

Both conditions are fulfilled, so given series is conditionally convergent.

Thus, the domain of convergence is $x \in [-2; 6)$.

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Exercises

In the exercises 343–351 determine if the following series are absolutely convergent, conditionally convergent or divergent:

343. $\sum_{n=1}^{\infty}\frac{x^n}{n^2}$. 344. $\sum_{n=1}^{\infty}\frac{3^n(x+1)^n}{n}$. 345. $\sum_{n=1}^{\infty}\frac{(x-5)^n}{5^n}$.

346. $\sum_{n=0}^{\infty}\frac{(x+2)^n}{3n+7}$. 347. $\sum_{n=1}^{\infty}\frac{2^n x^n}{n!}$. 348. $\sum_{n=1}^{\infty}\frac{n(x-4)^n}{10^n}$.

349. $\sum_{n=0}^{\infty}\frac{(x+2)^n}{3n+7}$. 350. $\sum_{n=1}^{\infty}\frac{2^n x^n}{n!}$. 351. $\sum_{n=1}^{\infty}\frac{n(x-4)^n}{10^n}$.

7.6. Taylor and Maclaurin Series

Let the function $f(x)$ has derivatives of every order at the point $x=a$. Power series

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots=$$

$$=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the *Taylor Series* for $f(x)$ about $x=a$.

The n-th partial sum for the Taylor Series is caller the *n-th degree Taylor polynomial of* $f(x)$:

$$T_n(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The *remainder* is defined to be

$$R_n(x)=f(x)-T_n(x).$$

Thus the function $f(x)$ can be written as

$$f(x)=T_n(x)+R_n(x).$$

Theorem. Suppose that $f(x)=T_n(x)+R_n(x)$. Then if $\lim_{n\to\infty}R_n(x)=0$ for $x\in(a-R;a+R)$, then

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$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all $x\in(a-R;a+R)$.

If we are talking about the Taylor Series about $a=0$, we call the series a *Maclaurin Series* for $f(x)$:

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.$$

Maclaurin Series for main elementary functions

1. $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!},\ x\in(-\infty;+\infty).$

2. $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!},\ x\in(-\infty;+\infty).$

3. $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},\ x\in(-\infty;+\infty).$

4. $\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{n=0}^{\infty}x^n,\ x\in(-1;1).$

5. $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n},\ x\in(-1;1].$

6. $(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\cdots=$

$$=1+\sum_{n=1}^{\infty}\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}x^n,\ x\in(-1;1].$$

Example 7.9. Find the Maclaurin Series for $f(x)=xe^{2x}$.

Solution

We already have a Maclaurin Series for e^x. For e^{2x} we have:

$$e^{2x}=1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}=\sum_{n=0}^{\infty}\frac{2^n}{n!}x^n.$$

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So

$$f(x) = xe^{2x} = x\sum_{n=0}^{\infty}\frac{2^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{2^n}{n!}x^{n+1}.$$

The domain of convergence is $x \in (-\infty; +\infty)$.

Exercises

In the exercises 352–357 find the Maclaurin Series for each of the following functions. Determine the domain of convergence:

352. $\frac{\sin x^2}{x}$. 353. $\frac{1}{1+x}$. 354. $\frac{x}{2-3x}$.

355. $\ln(2+x^2)$. 356. $\cos^2 2x$. 357. $xe^{2\sqrt{x}}$.

In the exercises 358–360 find the Taylor Series for each of the following functions about given point. Determine the domain of convergence:

358. $\frac{2}{x}$, $x = 3$. 359. $\frac{1}{\sqrt{2+x}}$, $x = -1$. 360. $\sin 2x$, $x = \frac{\pi}{4}$.

361. Calculate the integral $\int_0^{0.2} x\cos x dx$ with precision $\alpha = 0{,}001$.

362. Calculate the integral $\int_0^{1/5} \frac{\sin x - x}{x^2} dx$ with precision $\alpha = 0{,}001$.

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