Nonlinear Dynamics and Applications. Vol. 13 (2006) 37 - 46

Solving a relativistic quasipotential equation for a sum of a nonlocal separable interactions

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Solving of the finite-difference quasipotential equation involving a total quasipotential simulating the interaction of two relativistic spinless particles of unequal masses is obtained. The total interaction consisting of the superposition of a local and a sum of a nonlocal separable quasipotentials is the spherically symmetric quasipotential and it admits one true bound state. The problem is investigated within the relativistic quasipotential approach to quantum field theory. Explicit expressions are ob-
tained for the additions of the phase shift and their properties are investigated, the conditions under which bound states may exist are determined and the Levinson theorem is generalized.

PACS numbers: 11.10.Lm
Keywords: quasipotential equation, nonlocal interaction

1. INTRODUCTION

The main advantage of nonlocal separable potentials is that the partial-wave t-matrix for such potentials has a very simple form, and this makes it possible to continue it directly off the energy shell. Just this property is of paramount importance in nuclear physics and in a many-body problems. In particular, nonlocal separable interactions were used in solving Faddeev equations to the three-body problem. This approach also proved to be fruitful in solving the nonrelativistic inverse problem [1–5]. However, it cannot be applied to essentially relativistic systems [6, 7]. For example, for systems consisting of light quarks, the contribution of relativistic corrections to the interaction Hamiltonian appears to be comparable with the main nonrelativistic term.

The quasipotential approach [8] has still remained one, of the efficient methods for a relativistic description of two-body systems [9–12]. In the present study, the problem of solving the finite-difference quasipotential equation with a total quasipotential is considered within the relativistic quasipotential approach to quantum field theory [13]. The total quasipotential simulating the interaction between two relativistic spinless particles of unequal masses ($m_1 \neq m_2$) is the superposition of a local and a sum of a nonlocal separable quasipotentials. Besides, we will consider that the total interaction admits the existence of one true bound state and it's the local part $w(\rho)$ is known and is in accord with experimental data at low energies. Restricting our consideration to the case of spherically symmetric forces, we therefore take the total interaction in the form

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$$V(\vec{\rho},\vec{\rho}';E_{q'}) \equiv V(\vec{\rho},\vec{\rho}') = w(\rho)\,\delta(\vec{\rho}'-\vec{\rho}) + \sum_{l=0}^{\infty}\sum_{m=1}^{M_l}(2l+1)\,\varepsilon_{lm}v_{lm}(\rho)\,v_{lm}(\rho')\,P_l\left(\frac{\vec{\rho}\cdot\vec{\rho}'}{\rho\rho'}\right), \quad (1)$$

where $P_l(z)$ is a Legendre function of the first kind; $\rho = |\vec{\rho}|$, $\rho' = |\vec{\rho}'|$; and $\varepsilon_{lm} = \pm 1$. In the system of units where $\hbar = c = 1$, the relativistic analog of the differential Schrödinger equation for the wave function $\Psi_{q'}(\vec{\rho})$ in the configuration representation for particles of unequal masses with a quasipotential (1) is then given by [14]

$$\frac{m'^2}{\mu}\left[\cosh\left(i\lambda'\frac{\partial}{\partial\rho}\right) + \frac{i\lambda'}{\rho}\sinh\left(i\lambda'\frac{\partial}{\partial\rho}\right) - \frac{\lambda'^2}{2\rho^2}\Delta_{\theta,\varphi}\exp\left(i\lambda'\frac{\partial}{\partial\rho}\right) - \cosh\chi'\right]\Psi_{q'}(\vec{\rho}) + \quad (2)$$

$$+\int d\vec{\rho}'V(\vec{\rho},\vec{\rho}')\,\Psi_{q'}(\vec{\rho}') = 0,$$

where $\Delta_{\theta,\varphi}$ is the angular part of the Laplace operator, $\lambda' = 1/m'$ is the Compton wavelength connected with the effective relativistic particle of mass $m' = \sqrt{m_1 m_2}$, and $\mu = m'^2/(m_1+m_2)$.

We note that, within of the given approach Eq.(2) describes scattering of an effective relativistic particle of a mass m' having a relative 3-momentum $\vec{q}'$ and the total particle energy $\sqrt{S_{q'}}$ in the c.m. frame being proportional to the energy $E_{q'}$ of the effective relativistic particle of mass m' [14], that is

$$\sqrt{S_{q'}} = (m'/\mu)\,E_{q'},\ E_{q'} = \sqrt{m'^2+\vec{q}'^2} = m'\cosh\chi', \quad (3)$$

where χ' is the rapidity of the effective particle.

Following [15], we expand the wave function $\Psi_{q'}(\vec{\rho})$ in partial waves as

$$\Psi_{q'}(\vec{\rho}) = \sum_{l=0}^{\infty}(2l+1)\,i^l\frac{\psi_l(\rho,\chi')}{\rho}P_l\left(\frac{\vec{q}'\cdot\vec{\rho}}{q'\rho}\right), \quad q' = |\vec{q}'|.$$

Equation (2) can then be recast into the form

$$\left[\nabla + \left(1+\frac{l(l+1)}{r^{(2)}}\right)\nabla^* - 2\cosh\chi' + W(r)\right]\psi_l(r,\chi') + \quad (4)$$

$$+\sum_{m=1}^{M_l}\varepsilon_{lm}V_{lm}(r)\int_0^{\infty}dr'V_{lm}(r')\,\psi_l(r',\chi') = 0,$$

where $\nabla = \exp(-id/dr)$, $\nabla^* = \exp(id/dr)$, $V_{lm}(r) = \sqrt{8\pi\lambda'\mu/m'^2}\,\rho v_{lm}(\rho)$,

$$r^{(2)} = r(r+i), \quad W(r) = 2\mu w(\rho)/m'^2,\ \rho = \lambda' r,\ \rho' = \lambda' r'.$$

Thus, the possibility of representing the total energy of two relativistic spinless particles of unequal masses in the c.m. frame as a quantity that is proportional to the energy of one effective relativistic particle of mass m' enables us to reduce, within this approach, the relativistic problem of two bodies having unequal masses to a one-body problem.

The present study is devoted to solving Eq.(4) with the boundary condition

$$\psi_l(0,\chi') = 0, \quad (5)$$

to obtaining the expression for the additions of the phase shift to investigating the conditions of existence bound states and to generalizing the Levinson theorem for a superposition of a local and a sum of a nonlocal separable quasipotentials. Besides, we will consider that the total interaction admits the existence of the only true bound state.

2. Wave function and the additions of the phase shift

In order that a unique solution to Eq.(4) with the boundary condition in (5) to have, the local quasipotential $W(r)$ and the components $V_{lm}(r)$ of the nonlocal separable quasipotential must satisfy the conditions

$$rW(r) \in L_1(0,\infty),\ rV_{lm}(r) \in L_1(0,\infty),\ m = 1,2,...,M_l. \tag{6}$$

This means that the regular solution $\varphi_l(r,\chi')$ of Eq.(4) at $\varepsilon_{lm} \equiv 0$ with the boundary condition $\varphi_l(0,\chi') = 0$ in the case where the local quasipotential $W(r)$ not admits the existence of bound states, will satisfy the orthogonality and completeness properties [16]

$$\int_0^\infty dr\varphi_l(r,\chi)\varphi_l^*(r,\chi') = \frac{\delta(\cosh\chi - \cosh\chi')}{d\rho_l(\cosh\chi)/d(\cosh\chi)},\quad \int_1^\infty d\rho_l(\cosh\chi)\varphi_l(r,\chi)\varphi_l^*(r',\chi) = \delta(r'-r), \tag{7}$$

where the spectral density is in this case given by

$$d\rho_l(\cosh\chi)/d(\cosh\chi) = \sinh^{-1}(\chi)\tau_l(\chi),\quad \tau_l(\chi) = (2/\pi)Q_l^2(\coth\chi)\left|F_l^W(\chi)\right|^{-2}, \tag{8}$$

$$E = E_q/m' = \cosh\chi \geq 1.$$

Here, $Q_l(z)$ is a Legendre function of the second kind, and $F_l^W(\chi)$ is the Jost function for the local quasipotential $W(r)$ and is related to the corresponding phase shift $\delta_l^W(\chi)$ by the equation

$$F_l^W(\chi) = \left|F_l^W(\chi)\right|\exp\left[-i\delta_l^W(\chi)\right].$$

The solution to Eq.(4) with the boundary condition in (5) will search by means of the iterations by using the integral transformation of the wave function defined by the wave-function of the proceding step. We consider on the first step of iteration the superposition of $W(r)$ and $\varepsilon_{l1}V_{l1}(r)V_{l1}(r')$. It follows that we must find the solution $\psi_l^{(1)}(r,\chi')$ to Eq.(4) at $n = 1$ satisfying the boundary condition in the form of (5). The properties (7) permit us to introduce the relativistic integral transformations

$$\tilde{\psi}_l^{(1)}(\chi',\chi) = \int_0^\infty dr\psi_l^{(1)}(r,\chi')\psi_l^{(0)*}(r,\chi),\ \tilde{V}_{l1}^{(0)}(\chi) = \int_0^\infty drV_{l1}(r)\psi_l^{(0)*}(r,\chi), \tag{9}$$

$$\psi_l^{(1)}(r,\chi') = \int_1^\infty d\rho_l^{(0)}(\cosh\chi)\tilde{\psi}_l^{(1)}(\chi',\chi)\psi_l^{(0)}(r,\chi),\ V_{l1}(r) = \int_1^\infty d\rho_l^{(0)}(\cosh\chi)\tilde{V}_{l1}^{(0)}(\chi)\psi_l^{(0)}(r,\chi),$$

where $d\rho_l^{(0)}(\cosh\chi)/d(\cosh\chi) \equiv d\rho_l(\cosh\chi)/d(\cosh\chi)$, and the function $\psi_l^{(0)}(r,\chi) \equiv \varphi_l(r,\chi)$ satisfies the properties (7). Using the results of article [16] we should then have

$$\psi_l^{(1)}(r,\chi') = \psi_l^{(0)}(r,\chi') + \frac{1}{2}\varepsilon_{l1}N_{l1}(\chi')P\int_1^\infty d\rho_l^{(0)}(\cosh\chi)\frac{\tilde{V}_{l1}^{(0)}(\chi)\psi_l^{(0)}(r,\chi)}{\cosh\chi' - \cosh\chi}, \tag{10}$$

$$N_{l1}(\chi') = \varepsilon_{l1}\tilde{V}_{l1}^{(0)*}(\chi')/\Phi_{l1}(\cosh\chi'),\ \tan\delta_l^{V_{l1}}(\chi') = -(\pi/2)\varepsilon_{l1}\sinh^{-1}(\chi')A_{l1}(\chi')/\Phi_{l1}(\cosh\chi'),$$

$$\Phi_{l1}(\cosh\chi') = \varepsilon_{l1} + \frac{1}{2}P\int_0^\infty d\chi\frac{|A_{l1}(\chi)|}{\cosh\chi - \cosh\chi'} \ , \ A_{l1}(\chi') = \varepsilon_{l1}\tau_l(\chi')\left|\tilde{V}_{l1}^{(0)}(\chi')\right|^2,$$

where P means the principal value. Besides, the asymptotic behaviour of the wave function $\psi_l^{(1)}(r,\chi')$ is

$$\psi_l^{(1)}(r,\chi') = \frac{|F_l^W(\chi')|}{Q_l(\coth\chi')}\left[\cos\delta_l^{V_{l1}}(\chi')\right]^{-1}\sin\left[r\chi' - \frac{\pi l}{2} + \delta_l^{(1)}(\chi')\right] + O\left(e^{-\pi r/4}\right) \ , \quad (11)$$

$r \longrightarrow \infty$.

Here, $\delta_l^{(1)}(\chi') = \delta_l^W(\chi') + \delta_l^{V_{l1}}(\chi')$ is the total phase shift corresponding to the first step of iteration, and $\delta_l^{V_{l1}}(\chi')$ is its the addition due to the component $V_{l1}(r)$ of the separable interaction, and besides $\delta_l^{V_{l1}}(\chi') \leq \pi$ (the true bound states are absent). Moreover, the function $\psi_l^{(1)}(r,\chi')$ will satisfy the orthogonality and completeness properties:

$$\int_0^\infty dr\psi_l^{(1)}(r,\chi)\psi_l^{(1)*}(r,\chi') = \frac{\delta(\cosh\chi - \cosh\chi')}{d\rho_l^{(1)}(\cosh\chi)/d(\cosh\chi)}, \quad (12)$$

$$\int_1^\infty d\rho_l^{(1)}(\cosh\chi)\psi_l^{(1)}(r,\chi)\psi_l^{(1)*}(r',\chi) = \delta(r'-r),$$

where the spectral density is now given by

$$\frac{d\rho_l^{(1)}(\cosh\chi)}{d(\cosh\chi)} = \frac{d\rho_l^{(0)}(\cosh\chi)}{d(\cosh\chi)}\left[\cos\delta_l^{V_{l1}}(\chi)\right]^2 \quad . \quad (13)$$

Therefore, the properties (12) enable us to continue the process of iterations.

Consider now the superposition of $W(r)$ and $\sum\limits_{m=1}^{n}\varepsilon_{lm}V_{lm}(r)V_{lm}(r')$ $(n = 1,2,...,M_l)$. In order to solve (4) with such an interaction, we will use the orthogonality and completeness properties for the wave function $\psi_l^{(n-1)}(r,\chi')$. These the properties has the following form:

$$\int_0^\infty dr\psi_l^{(n-1)}(r,\chi)\psi_l^{(n-1)*}(r,\chi') = \frac{\delta(\cosh\chi - \cosh\chi')}{d\rho_l^{(n-1)}(\cosh\chi)/d(\cosh\chi)}, \quad (14)$$

$$\int_1^\infty d\rho_l^{(n-1)}(\cosh\chi)\psi_l^{(n-1)}(r,\chi)\psi_l^{(n-1)*}(r',\chi) = \delta(r'-r) \ ,$$

where

$$\frac{d\rho_l^{(n-1)}(\cosh\chi)}{d(\cosh\chi)} = \frac{d\rho_l^{(0)}(\cosh\chi)}{d(\cosh\chi)}\prod_{m=1}^{n-1}\left[\cos\delta_l^{V_{lm}}(\chi)\right]^2, \quad n = 1,2,...,M_l. \quad (15)$$

Here $\delta_l^{V_{lm}}(\chi)$ is the addition of phase shift due to the component $V_{lm}(r)$ of the separable interaction, and besides $\left|\delta_l^{V_{lm}}(\chi)\right| \leq \pi$ $(m = 1,2,...,M_l - 1)$. In other words, at each step we use the integral transformations defined by $\psi_l^{(n-1)}(r,\chi)$. These the transformations has the following form:

$$\tilde{\psi}_l^{(n)}(\chi',\chi) = \int\limits_0^\infty dr \psi_l^{(n)}(r,\chi')\psi_l^{(n-1)^*}(r,\chi), \;\; \tilde{V}_{ln}^{(n-1)}(\chi) = \int\limits_0^\infty dr V_{ln}(r)\psi_l^{(n-1)^*}(r,\chi), \tag{16}$$

$$\begin{gathered}\psi_l^{(n)}(r,\chi') = \int\limits_1^\infty d\rho_l^{(n-1)}(\cosh\chi)\tilde{\psi}_l^{(n)}(\chi',\chi)\psi_l^{(n-1)}(r,\chi), \\ V_{ln}(r) = \int\limits_1^\infty d\rho_l^{(n-1)}(\cosh\chi)\tilde{V}_{ln}^{(n-1)}(\chi)\psi_l^{(n-1)}(r,\chi), \;\; n = 1,2,...,M_l.\end{gathered} \tag{17}$$

By applying the transformations (17) to Eq.(4) with such an interaction, we then obtain

$$(\cosh\chi' - \cosh\chi)\tilde{\psi}_l^{(n)}(\chi',\chi) = \frac{1}{2}\varepsilon_{ln}N_{ln}(\chi')\tilde{V}_{ln}^{(n-1)}(\chi), \tag{18}$$

where

$$\begin{gathered}N_{ln}(\chi') = \int\limits_0^\infty dr' V_{ln}(r')\psi_l^{(n)}(r',\chi') = \int\limits_1^\infty d\rho_l^{(n-1)}(\cosh\chi)\tilde{\psi}_l^{(n)}(\chi',\chi)\tilde{V}_{ln}^{(n-1)^*}(\chi), \\ n = 1,2,...,M_l.\end{gathered} \tag{19}$$

Now note that by virtue of the conditions in (6) we have the asymptotic expression for the wave function $\psi_l^{(n)}(r,\chi')$ in the form

$$\psi_l^{(n)}(r,\chi') = \frac{|F_l^W(\chi')|}{Q_l(\coth\chi')}\prod_{m=1}^{n}\left[\cos\delta_l^{V_{lm}}(\chi')\right]^{-1}\sin\left[r\chi' - \frac{\pi l}{2} + \delta_l^{(n)}(\chi')\right] + O\left(e^{-\pi r/4}\right), \tag{20}$$
$r \to \infty$,

where $\delta_l^{(n)}(\chi') = \delta_l^W(\chi') + \sum\limits_{m=1}^{n}\delta_l^{V_{lm}}(\chi')$ is the total phase shift $(n = 1,2,...,M_l)$. Besides, the function $\tilde{V}_{ln}^{(n-1)}(\chi)$ is everywhere continuous and that the function $Q_l(\coth\chi)\prod\limits_{m=1}^{n-1}\cos\left[\delta_l^{V_{lm}}(\chi)\right]\times$
$\times\tilde{V}_{ln}^{(n-1)}(\chi)\left|F_l^W(\chi)\right|^{-1}$ is differentiable for all $\chi \geq 0$. Moreover, we find from (16) that

$$\begin{gathered}Q_l(\coth\chi)\prod_{m=1}^{n-1}\cos\left[\delta_l^{V_{lm}}(\chi)\right]\,\tilde{V}_{ln}^{(n-1)}(\chi)\left|F_l^W(\chi)\right|^{-1} = O(1), \;\; |\chi| \to \infty, \\ \tilde{V}_{ln}^{(n-1)}(\chi) = O(1), \;\; \chi \to 0,\end{gathered} \tag{21}$$

provided that the conditions in (6) hold.

For scattering states ($E' = \cosh\chi' \geq 1$), the solution to Eq.(4) is given by

$$\tilde{\psi}_l^{(n)}(\chi',\chi) = \frac{\delta(\cosh\chi - \cosh\chi')}{d\rho_l^{(n-1)}(\cosh\chi)/d(\cosh\chi)} + \frac{1}{2}\varepsilon_{ln}N_{ln}(\chi')P\frac{\tilde{V}_{ln}^{(n-1)}(\chi)}{\cosh\chi' - \cosh\chi}, \tag{22}$$

where the factor in front of the δ-function was chosen in accordance with the normalization of the wave function, that is at $\varepsilon_{ln} \equiv 0$ the representation in (17) for $\psi_l^{(n)}(r,\chi')$ must lead to the solution $\psi_l^{(n-1)}(r,\chi')$ $(n = 1,2,...,M_l)$. Substituting the solution in (22) into the representation in (17) for $\psi_l^{(n)}(r,\chi')$ and (19) we obtain

$$\psi_l^{(n)}(r,\chi') = \psi_l^{(n-1)}(r,\chi') + \frac{1}{2}\varepsilon_{ln}N_{ln}(\chi')P\int\limits_1^\infty d\rho_l^{(n-1)}(\cosh\chi)\frac{\tilde{V}_{ln}^{(n-1)}(\chi)\psi_l^{(n-1)}(r,\chi)}{\cosh\chi' - \cosh\chi}, \tag{23}$$

$$N_{ln}(\chi') = \varepsilon_{ln}\tilde{V}_{ln}^{(n-1)*}(\chi')/\Phi_{ln}(\cosh\chi'), \tag{24}$$

where

$$\Phi_{ln}(\cosh\chi') = \varepsilon_{ln} + \frac{1}{2}P\int\limits_0^\infty d\chi\frac{|A_{ln}(\chi)|}{\cosh\chi - \cosh\chi'}, \tag{25}$$

$$A_{ln}(\chi) = \varepsilon_{ln}\tau_l(\chi)\prod_{m=1}^{n-1}\left[\cos\delta_l^{V_{lm}}(\chi)\right]^2\left|\tilde{V}_{ln}^{(n-1)}(\chi)\right|^2,\ n = 1,2,...,M_l. \tag{26}$$

The principal values of the integrals in (23) and (25) exist since the function $\tilde{V}_{ln}^{(n-1)}(\chi)$ is differentiable and since, by virtue of the conditions in (21), these integrals converge at both limits.

By using the asymptotic expression in (20) for $\psi_l^{(n-1)}(r,\chi')$, the solution in (23) represent in the form

$$\begin{aligned}\psi_l^{(n)}(r,\chi') = &\frac{|F_l^W(\chi')|}{Q_l(\coth\chi')}\prod_{m=1}^{n-1}\left[\cos\delta_l^{V_{lm}}(\chi')\right]^{-1}\sin\left[r\chi' - \frac{\pi l}{2} + \delta_l^{(n-1)}(\chi')\right] - \\ &-\varepsilon_{ln}N_{ln}(\chi')P\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}d\chi\frac{Q_l(\coth\chi)\tilde{V}_{ln}^{(n-1)}(\chi)}{|F_l^W(\chi)|(\cosh\chi - \cosh\chi')}\prod_{m=1}^{n-1}\left[\cos\delta_l^{V_{lm}}(\chi)\right]\times \\ &\times\exp\left[i\left(r\chi - \frac{\pi l}{2} + \delta_l^{(n-1)}(\chi)\right)\right] + O\left(e^{-\pi r/4}\right),\ r\to\infty.\end{aligned}$$

The principal value of the integral in the last equality can easily be calculated for $r\to\infty$ if we use the relation

$$\frac{1}{\alpha - i\eta} = i\pi\delta(\alpha) + P\left(\frac{1}{\alpha}\right),\ \eta\to +0,$$

and then apply the residue theorem in performing integration along the boundary of the region $0 \le \mathrm{Im}\chi \le \pi/2$. The result is

$$\begin{aligned}\psi_l^{(n)}(r,\chi') = &\frac{|F_l^W(\chi')|}{Q_l(\coth\chi')}\prod_{m=1}^{n-1}\left[\cos\delta_l^{V_{lm}}(\chi')\right]^{-1}\sin\left[r\chi' - \frac{\pi l}{2} + \delta_l^{(n-1)}(\chi')\right] - \\ &-\frac{\varepsilon_{ln}N_{ln}(\chi')Q_l(\coth\chi')}{\sinh\chi'\,|F_l^W(\chi')|}\prod_{m=1}^{n-1}\left[\cos\delta_l^{V_{lm}}(\chi')\right]\cos\left[r\chi' - \frac{\pi l}{2} + \delta_l^{(n-1)}(\chi')\right] + \\ &+O\left(e^{-\pi r/4}\right),\ r\to\infty.\end{aligned} \tag{27}$$

Finally, taking into account the expressions (24) – (26), we set

$$\tan\delta_l^{V_{ln}}(\chi') = -(\pi/2)\varepsilon_{ln}\sinh^{-1}(\chi')A_{ln}(\chi')/\Phi_{ln}(\cosh\chi'),\ n = 1,2,...,M_l. \tag{28}$$

The asymptotic behaviour of the wave function in (27) is then given by the expression in (20).

3. Bound states and Levinson theorem

Suppous that there exists at least one bound state at energy $E^{(n)} = \cosh\chi^{(n)} \geq 0$ ($n = 1, 2, ..., M_l$). The solution of Eq. (18) has then the form

$$\tilde{\psi}_l^{(n)}\left(\chi^{(n)}, \chi\right) = -\frac{1}{2}\varepsilon_{ln} N_{ln}\left(\chi^{(n)}\right) P\frac{\tilde{V}_{ln}^{(n-1)}\left(\chi^{(n)}\right)}{\cosh\chi - E^{(n)}}, \quad n = 1, 2, ..., M_l. \tag{29}$$

The substitution of the solutions in (29) into to equality in (19) leads to an equation for eigenvalues

$$\Phi_{ln}\left(E^{(n)}\right) = \varepsilon_{ln} + \frac{1}{2}P\int_0^\infty d\chi \frac{|A_{ln}(\chi)|}{\cosh\chi - E^{(n)}} = 0, \quad n = 1, 2, ..., M_l. \tag{30}$$

Eq. (30) may have solutions at $\varepsilon_{ln} = \pm 1$ for the spurious bound states associated with the component $V_{ln}(r)$ of the separable interaction whose energies $E_{fk}^{(n)}$ satisfies the condition

$$E_{fk}^{(n)} = \cosh\chi_{fk}^{(n)} \geq 1, \ k = \begin{cases} 0, 1, ..., \nu_l^{(n)} - 1, \ \varepsilon_{ln} = 1, \\ 1, 2, ..., \nu_l^{(n)}, \ \varepsilon_{ln} = -1, \ n = 1, 2, ..., M_l. \end{cases} \tag{31}$$

At the same time, from Eq. (30) it follows that the values of $\varepsilon_{ln} = -1$ corresponds to the true bound states of the total interaction whose energies $E_t^{(n)}$ lies in the range

$$0 \leq E_t^{(n)} = \cosh\chi_t^{(n)} < 1, \ \chi_t^{(n)} = i\kappa_t^{(n)}, \ 0 < \kappa_t^{(n)} \leq \pi/2, \ n = 1, 2, ..., M_l. \tag{32}$$

However, the expressions in (14) and (15) will have the place provided the true bound states are absent at $n = 1, 2, ..., M_l - 1$, whereas at $n = M_l$ it may be. This is so provided that

$$\frac{2}{\pi}\int_0^\infty d\chi \prod_{m=1}^{n-1}\left[\cos\delta_l^{V_{lm}}(\chi)\right]^2 \left|\tilde{V}_{ln}^{(n-1)}(\chi)/F_l^W(\chi)\right|^2 < 1 \tag{33}$$

at $\varepsilon_{ln} = -1$, $n = 1, 2, ..., M_l - 1$ (the true bound states are absent), wheares

$$\frac{2}{\pi}\int_0^\infty d\chi \prod_{m=1}^{M_l-1}\left[\cos\delta_l^{V_{lm}}(\chi)\right]^2 \left|\tilde{V}_{lM_l}^{(M_l-1)}(\chi)/F_l^W(\chi)\right|^2 > 1 \tag{34}$$

at $\varepsilon_{lM_l} = -1$ (there exists the only true bound state at $n = M_l$). The last conditions is associated with the fact that, for any $l, \chi \geq 0$, the function $g_l(\chi)$ is bounded, that is

$$g_l(\chi) = \frac{1}{2}\frac{Q_l^2(\coth\chi)}{\cosh\chi - E_t^{(n)}} \leq \max g_l(\chi) \approx \frac{\pi(\tanh\chi_{\max})^{2l}}{4^{l+1}\cosh\chi_{\max}}\left[1 - \frac{l+1}{2l+3}\tanh^2\chi_{\max}\right] < 1.$$

For the case of spurious bound states at energies (31) the asymptotic behaviour of the wave function takes the form (27), where the first term is omitted, that is

$$\psi_l^{(n)}(r, \chi_{fk}^{(n)}) = -\frac{\varepsilon_{ln}N_{ln}(\chi_{fk}^{(n)})Q_l(\coth\chi_{fk}^{(n)})\tilde{V}_{ln}^{(n-1)}(\chi_{fk}^{(n)})}{\sinh\chi_{fk}^{(n)}\left|F_l^W(\chi_{fk}^{(n)})\right|}\prod_{m=1}^{n-1}\cos\left[\delta_l^{V_{lm}}(\chi_{fk}^{(n)})\right] \times$$
$$\times \cos\left[r\chi_{fk}^{(n)} - \frac{\pi l}{2} + \delta_l^{(n-1)}(\chi_{fk}^{(n)})\right] + O\left(e^{-\pi r/4}\right), \ r \to \infty, \ n = 1, 2, ..., M_l.$$

From this relation, it follows that the wave function $\psi_l^{(n)}(r,\chi_{fk}^{(n)})$ asymptotically tends to zero, provided that

$$\tilde{V}_{ln}^{(n-1)}\left(\chi_{fk}^{(n)}\right)=0,\ n=1,2,...,M_l. \tag{35}$$

Since the boundary condition (5) is also satisfied, spurious bound states associated with the component $V_{ln}(r)$of the separable interaction correspond to the energies in (31). Moreover, fulfillment of the conditions in (30) and (35) means that, at the energy values in (31), the phase-shift additions $\delta_l^{V_{ln}}(\chi')$ decreases with increasing χ', passing through the values πk ($k=\{0,1,...,\nu_l^{(n)}-1,\,\varepsilon_{ln}=1;\quad 1,2,...,\nu_l^{(n)},\quad \varepsilon_{ln}=-1;\,n=1,2,...,M_l\}$). This is because both the numerator and the denominator on the right-hand side of the equality in (28) vanish at these energy values by virtue of the conditions in (30) and (35). But it follows from the definitions in (25) and (26) that the functions $\Phi_{ln}(\cosh\chi')$ and $A_{ln}(\chi')$ exist and are differentiable. Moreover, the function $A_{ln}(\chi')$ has a zero of at least the second order at the points $\chi'=\chi_{fk}^{(n)}$, while the function $\Phi_{ln}(\cosh\chi')$ has, at these points, only a simple zero since

$$\left.\frac{d\Phi_{ln}(\cosh\chi')}{d\chi'}\right|_{\chi'=\chi_{fk}^{(n)}}=\frac{1}{2}\sinh\chi_{fk}^{(n)}\int\limits_0^\infty d\chi\frac{|A_{ln}(\chi)|}{\left(\cosh\chi-\cosh\chi_{fk}^{(n)}\right)^2}>0.$$

This means that, at $\chi'=\chi_{fk}^{(n)}$, the quantity $\tan\delta_l^{V_{ln}}(\chi')$ vanishes and changes sign, that is

$$\delta_l^{V_{ln}}\left(\chi_{fk}^{(n)}\right)=\pi k,\ k=\begin{cases}0,1,...,\nu_l^{(n)}-1,\ \varepsilon_{ln}=1,\\ 1,2,...,\nu_l^{(n)},\ \varepsilon_{ln}=-1,\end{cases}\left.\frac{d\delta_l^{V_{ln}}(\chi')}{d\chi'}\right|_{\chi'=\chi_{fk}^{(n)}}<0,\quad n=1,2,...,M_l.$$

If the denominator on the right-hand side of (28) does not vanish at $\chi'=\chi_{fk}^{(n)}$, the additions of the phase shift will only touch the straight lines $\delta_l^{V_{ln}}=\pi k$ (k is an integer) from above or from below, but it will not intersect them. Besides, studying the behaviour of $\delta_l^{V_{ln}}(\chi')$ as a function of χ', one can obtain the values of the energies $E_{fk}^{(n)}$ at which spurious bound states exist. At the same time, the values of ε_{ln} one can determine by the sign of the phase-shift additions $\delta_l^{V_{ln}}(\chi')$ at high energies ($\chi'\to+\infty$). By using the astimate in (21) and expression (28), we finally find that $\tan\delta_l^{V_{ln}}(\infty)=0$. This means that, we can choose the function $\delta_l^{V_{ln}}(\chi')$ in such a way as to ensure fulfillment of the condition

$$\delta_l^{V_{ln}}(\infty)=0,\ n=1,2,...,M_l. \tag{36}$$

Let us now consider the conditions of existence the only true bound state with energy in (32) associated with the total interaction ($n=M_l$). The energy $E_l^{(M_l)}$ of given the only true bound state will determine as the root of equation (30) provided that the conditions in (33) and (34) are satisfied. Obviously, the boundary condition (5) is satisfied for given state, and that its wave function asymptotically tends to zero for $r\to\infty$. This can be proven by substituting the solution in (29) into transformation (17) for $\psi_l^{(n)}(r,\chi')$ at $n=M_l$, by using the asymptotic behaviour of the wave function in (20) for $\psi_l^{(M_l-1)}(r,\chi')$, and there upon performing integration along the boundary of the region $0\le\mathrm{Im}\chi\le\pi/2$ by means of the residue theorem. As a result, we have

$$\psi_l^{(M_l)}(r,\chi_t^{(M_l)})=O\left(\exp\left[-r\min\left(\kappa_t^{(M_l)},\ \pi/4\right)\right]\right),\ r\to\infty.$$

In order to obtain the Levinson theorem, we will use the results reported in [15, 16]. Taking into consideration the condition (36) and usual Levinson theorem for a local quasipotential that does not admit bound states, that is

$$\delta_l^W(0) - \delta_l^W(\infty) = \delta_l^W(0) = 0,$$

we then obtain

$$\delta_l^{V_{ln}}(0) = \pi(\sigma_l^{(n)} - \sigma_l^{(n-1)} + \nu_l^{(n)}), \quad n = 1, 2, ..., M_l. \tag{37}$$

Here $\nu_l^{(n)}$ is the number of spurious bound states associated with the component $V_{ln}(r)$ of the separable interaction at the energies in (31), and $\sigma_l^{(n)}$ is the number of true bound states of the total interaction at the energies in (32), moreover

$$\sigma_l^{(n)} = \begin{cases} 0, \ \varepsilon_{ln} = 1, \ n = 1, 2, ..., M_l; \\ 0, \ \varepsilon_{ln} = -1, \ n = 1, 2, ..., M_l - 1; \\ 1, \ \varepsilon_{lM_l} = -1, \ n = M_l. \end{cases}$$

4. Conclusion

Solving of the finite-difference quasipotential equation involving a total quasipotential simulating the interaction of two relativistic spinless particles of unequal masses is obtained. The total interaction consisting of the superposition of a local quasipotential and a sum of nonlocal separable quasipotentials is the spherically symmetric quasipotential and it admits the only true bound state. The problem is investigated within the relativistic quasipotential approach to quantum field theory. Besides, the local component of the total interaction is supposed to be known and that it is in accord with experimental data at low energies. The given method is directly associated with the orthogonality and completeness properties for the partial wave function of the local quasipotential. It has been shown that the orthogonality and completeness properties for the partial wave function associated with the superposition of a local quasipotential and a sum a nonlocal separable quasipotentials are also satisfied and this has provided us the process of iterations. This has permitted us to find an explicit expressions for the phase-shift additions and to investigate their properties, to determine the conditions under which the true and spurious bound states may exist, and to generalize the Levinson theorem.

Acknowledgments

I am grateful to Yu.S. Vernov, I.L. Solovtsov, A.M. Shirokov and V.V. Andreev for interest in this study and enlightening discussions on the results presented here.

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