Leptonic Widths in the Relativistic Quasipotential Approach: the Case of Arbitrary Masses

Yu. Chernichenko *, *O. Solovtsova* †

International Center for Advanced Studies, Gomel State Technical University, Belarus

Abstract

The new relativistic leptonic decay widths in quantum chromodynamics are obtained. Consideration is conducted within the framework of completely covariant of the quasipotential approach in quantum field theory, formulated in the relativistic configurational representation in the case of two particles of arbitrary masses.

1 Introduction

In the two-particle approximation the square of the Bethe-Salpeter (BS) amplitude of two charged particles $\chi_{\rm BS}(x)$ at $x = (\mathbf{r}, \tau) = 0$ and hence at the relative time $\tau = 0$ is an important quantity for $q\bar{q}$ systems. For example, it appears in the expressions for the Drell ratio $R(s)$, for the leptonic and hadronic widths $\Gamma(e^+e^-)$ and $\Gamma(3g)$ for the 1^- states, and the hadronic widths $\Gamma(2g)$ for the 0^- states. The leptonic widths $\Gamma(e^+e^-)$ for the decay of 1^- states ($\ell = 0$), which will consider in this paper, is [1]–[7]

$$\Gamma(e^+e^-) = 16\pi\alpha^2 e_q^2 \frac{|\chi_{\rm BS}(x=0)|^2}{M^2}, \tag{1}$$

where α is the fine-structure constant, e_q is the quark charge in the units of e, and M is the total c.i.s. energy of the interacting particles ($q\bar{q}$ or e^+e^- system) with arbitrary masses m_1 and m_2.

*E-mail: chern@gstu.by, chyud@mail.ru

†E-mail: solovtsova@gstu.by, olsol@theor.jinr.ru

Our aim here is to generalize the method proposed in [8] for obtaining relativistic leptonic decay widths for value of the orbital moment $\ell = 0$. Consideration is conducted within the framework of completely covariant of the relativistic quasipotential (RQP) approach in quantum field theory (QFT) proposed by Logunov and Tavkhelidze [9] in the form proposed in [10] for the case of two particles of arbitrary masses m_1 and m_2.

2 Background

Representing the square of the BS amplitude in zero, $|\chi_{\rm BS}(x = 0)|^2$, through square of the wave function in zero for the $\ell = 0$ state, $|\psi_0^{\rm nr}(0, E_n)|^2$, corresponding to the nonrelativistic Schrödinger equation in the case of two particles of equal masses with the confining potential $V_{\rm c}(r)$ leads in the JWKB approximation to formula [11]–[13] (here $\hbar = c = 1$)

$$|\psi_0^{\rm nr}(0, E_n)|^2 = \frac{m_q^2}{4\pi^2} v_n^{\rm nr} \frac{dE_n}{dn}, \tag{2}$$

where $v_n^{\rm nr} = \sqrt{E_n/m_q}$ is the nonrelativistic velocity of free quark with mass m_q and kinetic energy $E_n/2$ for given of level n, and the confining potential $V_{\rm c}(r)$ for simplicity is considered equal zero at $r = 0$ ($V_{\rm c}(0) = 0$).

Generalization of the nonrelativistic expression (2) by adding to the confining potential $V_{\rm c}(r)$ the Coulomb interaction $V_{\rm s}(r) = -4\alpha_{\rm S}/3r$, where $\alpha_{\rm S}$ is the strong coupling constant and the confining potential $V_{\rm c}(r)$ is considered again for simplicity equal zero at $r = 0$ was executed in [14], [15]. Their result as under $E_n > 0$, so and under $E_n < 0$ is

$$|\psi_0^{\rm nr}(0, E_n)|^2 = F(v_n^{\rm nr}) \frac{m_q^2}{4\pi^2} v_n^{\rm nr} \frac{dE_n}{dn}, \tag{3}$$

where expression

$$F(v) = \frac{4\pi\alpha_{\rm S}}{3v} \left[1 - \exp\left(-\frac{4\pi\alpha_{\rm S}}{3v}\right)\right]^{-1} \tag{4}$$

is the Coulomb S-factor.

In Refs. [16], [17] and also in the case of two particles of equal masses was obtained the relativistic analog of formula (2) for the Salpeter wave function $\Psi_0^{\rm rel}(r, M_n)$ ($\ell = 0$) for an instantaneous $q\bar{q}$ interaction in the form

$$|\Psi_0^{\rm rel}(0, M_n)|^2 = \frac{M_n^2}{16\pi^2} v_n^{\rm rel} \frac{dM_n}{dn}, \tag{5}$$

where $v_n^{\rm rel} = \sqrt{1 - 4m_q^2/M_n^2}$ is the relativistic velocity of a free quark with mass m_q and the total energy $M_n/2 = (2m_q + E_n)/2$. The existence of such a relation for the relativistic case was earlier offered but is not proved in Ref. [18].

Generalization of the nonrelativistic expression (3) was executed in Ref. [1]. Their method is based on replacement of the full BS interaction kernel by an appropriate instantaneous interaction (the Salpeter approximation). In terms of a first approximation they obtained ($M_n \lesssim 2m_q$)

$$|\chi_{\rm BS}(x=0)|^2|_{\ell=0} = |\Psi_0^{\rm rel}(0, M_n)|^2 = F(v_n^{\rm rel})\frac{M_n^2}{16\pi^2}v_n^{\rm rel}\frac{dM_n}{dn}, \tag{6}$$

where the Coulomb S-factor is determined in Eq. (4).

In the RQP approach [9] for the case of two particles of arbitrary masses m_1 and m_2 [10] the BS amplitude in zero $\chi_{\rm BS}(x = 0)$ can be expressed through the RQP wave functions in the momentum space $\Psi_{q'}(\mathbf{p}')$ with a relative three-momentum $\mathbf{p}'$ and the $\mathbf{r}$-representation $\psi_{q'}(\mathbf{r})$ as

$$\chi_{\rm BS}(x=0) = \frac{1}{(2\pi)^3}\int d\Omega_{\mathbf{p}'}\Psi_{q'}(\mathbf{p}') = \lim_{r\to i\lambda'}\psi_{q'}(\mathbf{r}), \tag{7}$$

where $d\Omega_{\mathbf{p}'} = m'c^2 d\mathbf{p}'/E_{p'}$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid $E_{p'}^2 - c^2\mathbf{p}'^2 = m'^2c^4$, $m' = \sqrt{m_1 m_2}$, $\lambda' = \hbar/m'c$.

In the RQP approach the formula (1) must be modified. The relativistic modification of the formula (1) in RQP for the case of interaction between two relativistic particles of equal masses ($m_1 = m_2 = m$) by quasipotential

$$V(r) = -\frac{\kappa_{\rm S}^2}{r} + \sigma r^s,\ \sigma, s > 0, \tag{8}$$

performed in [8]. The method developed there is based on the RQP approach in [19] and uses the possibility of passing from an integral equation in the three-dimensional Lobachevsky momentum space to a finite-difference equation in the three-dimensional relativistic configuration representation introduced in [20] for the interaction of two relativistic equal-mass particles. Their result is

$$\Gamma_l(\mu \to e^+e^-) = \frac{16\pi\alpha^2 e_q^2}{M_n^2}\left|\psi_{q_n'}(0)\right|^2\Big|_{\ell=0} =$$

$$= \alpha^2 e_q^2 \frac{Zm\alpha_{\rm S}}{\pi\hbar^3 M_n^2}\frac{dM_n}{dn}\left|F\left(1 - \frac{\alpha_{\rm S}}{2\sin\kappa_n}, 1; 2; 1 - e^{-2i\kappa_n}\right)\right|^2, \tag{9}$$

where $\alpha_S = \kappa_S^2/\hbar c$, $\kappa_n = \arccos(M_n/2mc^2)$, $F(a,b;c;z)$ is the hypergeometric function.

3 Finite-difference form of quasipotential equation

The basis of our consideration is completely covariant RQP Schrödinger equation into the **r**-representation in terms finite differences constructed in [10] for the RQP wave function $\psi_{q'}(\mathbf{r})$ of two relativistic particles of arbitrary masses. For spherically symmetric potentials this equation is

$$\left(2E_{q'} - \widehat{H}_0\right)\psi_{q'}(\mathbf{r}) = \frac{2\mu}{m'}V(r;E_{q'})\psi_{q'}(\mathbf{r}), \tag{10}$$

where $\mu = m_1 m_2/(m_1+m_2)$ is the usual reduced mass, the operator

$$\widehat{H}_0 = 2m'c^2\left[\cosh\left(i\lambda'\frac{\partial}{\partial r}\right) + \frac{i\lambda'}{r}\sinh\left(i\lambda'\frac{\partial}{\partial r}\right) - \frac{\lambda'^2}{2r^2}\Delta_{\theta,\varphi}\exp\left(i\lambda'\frac{\partial}{\partial r}\right)\right]$$

is the operator of free Hamiltonian while $\Delta_{\theta,\varphi}$ is its the angular part, and $\lambda' = \hbar/m'c$ is the Compton wavelengths of the effective relativistic particle with mass m', a relative three-momentum $\mathbf{q}'$ and the energy $E_{q'} = c\sqrt{m'^2c^2 + \mathbf{q}'^2}$, emerging instead of the system of two particles and carrying the total c.i.s. energy $M = \sqrt{s}$ of the interacting particles proportional to the energy $E_{q'}$ (see [10, 21]):

$$M = \sqrt{s} = c\sqrt{m_1^2c^2 + \mathbf{q}^2} + c\sqrt{m_2^2c^2 + \mathbf{q}^2} = \frac{m'}{\mu}E_{q'}.$$

The quasipotential $V(r;E_{q'})$ is local in the sense of the Lobachevsky geometry and in general depends parametrically on the energy $E_{q'}$, and the module of the radius-vector, $\mathbf{r}$ ($\mathbf{r} = r\mathbf{n}$, $|\mathbf{n}| = 1$), is a relativistic invariant [10], [21], [22].

By using the expansion of the RQP wave function $\psi_{q'}(\mathbf{r})$ on a Legendre function $P_\mu(z)$ of the first kind

$$\psi_{q'}(\mathbf{r}) = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell\frac{\varphi_\ell(r,\chi)}{r}P_\ell\left(\frac{\mathbf{q}'\cdot\mathbf{r}}{q'r}\right), \tag{11}$$

Eq. (10) transformed to the form

$$\left[\cosh\left(i\lambda'\frac{d}{dr}\right)+\frac{\lambda'^2\ell(\ell+1)}{2r(r+i\lambda')}\exp\left(i\lambda'\frac{d}{dr}\right)-X(r)\right]\varphi_\ell(r,\chi)=0. \quad (12)$$

Here

$$X(r)=\frac{\mu}{m'^2c^2}\left(M-V(r,\chi)\right), \quad (13)$$

and χ is the rapidity which are related to three-momentum $\mathbf{q}'$ and the energy $E_{q'}$ by

$$\mathbf{q}'=m'c\sinh\chi\mathbf{n}_{q'},\ |\mathbf{n}_{q'}|=1,\ E_{q'}=m'c^2\cosh\chi,\ M=\frac{m'^2c^2}{\mu}\cosh\chi. \quad (14)$$

4 Relativistic analog of the modified JWKB method and leptonic decay widths

We will seek the JWKB solution of RQP Eq. (12) in the usual form [8]

$$\varphi_\ell(r,\chi)=\exp\left[\frac{i}{\hbar}g(r)\right],$$
$$g(r)=g_0(r)+\frac{\hbar}{i}g_1(r)+\left(\frac{\hbar}{i}\right)^2g_2(r)+\dots. \quad (15)$$

For first two members of the decomposition in (15) the JWKB solutions with the left r_{L} and the right r_{R} of the classical turning points in the inner region $r_{\mathrm{L}}\le r\le r_{\mathrm{R}}$ are given then as

$$\varphi_\ell^{\mathrm{L,R}}(r,\chi)=\frac{C_{\mathrm{L,R}}}{2\sqrt[4]{X^2(r)-R^2(r)}}\left\{\exp\left[i\alpha_+^{\mathrm{L,R}}(r)\mp\frac{i\pi}{4}\right]+\right.$$
$$\left.+\exp\left[i\alpha_-^{\mathrm{L,R}}(r)\pm\frac{i\pi}{4}\right]\right\}, \quad (16)$$

where

$$R(r)=\sqrt{1+\frac{\Lambda^2}{r^2}},\Lambda=\lambda'\left(\ell+\frac{1}{2}\right),\alpha_\pm^{\mathrm{L,R}}(r)=\frac{1}{\lambda'}\int_{r_{\mathrm{L,R}}}^{r}dt\chi_\pm(t),$$
$$\chi_\pm(r)=\ln\left[X(r)\pm\sqrt{X^2(r)-R^2(r)}\right], \quad (17)$$

and $C_{\rm L,R}$ are the normalization constants. The turning points $r_{\rm L,R}$ are branch points of root in the JWKB solutions (16) that leads to equation

$$X^2(r) - R^2(r) = 0. \tag{18}$$

The JWKB quantization condition, either as in the nonrelativistic case, find from condition of the coincidence of waves functions in Eqs. (16) in point $r \in (r_{\rm L}; r_{\rm R})$ for what necessary to choose

$$C_{\rm L} = C_0 \exp\left[-\frac{i}{\lambda'}\int\limits_{r_{\rm L}}^{r} dt \ln R(t)\right], C_{\rm R} = C_0(-1)^n \exp\left[-\frac{i}{\lambda'}\int\limits_{r_{\rm R}}^{r} dt \ln R(t)\right],$$

where C_0 is an arbitrary constant. This leads to the JWKB quantization condition

$$\int\limits_{r_{\rm L}}^{r_{\rm R}} dr\,[\chi_+(r) - \ln R(r)] = \pi\lambda'\left(n + \frac{1}{2}\right),\ n = 0, 1, \ldots, \ell \geq 0. \tag{19}$$

Condition of applicability of the relativistic JWKB method has the form

$$\lambda'\left|\frac{\cosh\chi_{\rm eff}(r)}{\chi_+(r)\sinh\chi_{\rm eff}(r)}\frac{d\chi_+(r)}{dr}\right| \ll 1,\ \ell \geq 0, \tag{20}$$

where $\chi_{\rm eff}(r) = {\rm arcosh}X_{\rm eff}(r)$, $X_{\rm eff}(r) = \cosh\chi_{\rm eff}(r) = X(r)/R(r)$. In the case of $\ell = 0$ condition (20) is converted in inequality

$$\lambda'\left|\frac{\cosh\chi(r)}{\chi(r)\sinh\chi(r)}\frac{d\chi(r)}{dr}\right| \ll 1, \tag{21}$$

where

$$\chi(r) = {\rm arcosh}X(r) = \ln[X(r) + \sqrt{X^2(r) - 1}] \tag{22}$$

is the rapidity of effective relativistic particle of the mass m' that moves in the field of potential $V(r)$, but function $X(r)$ is determined in (13). In the nonrelativistic limit under $\chi \to 0$ and $M = m'^2c^2/\mu + E$, the inequality in (21) take then the usual form $|d\lambda'_{\rm nr}(r)/dr| \ll 1$, $\lambda'_{\rm nr}(r) = \hbar/p_{\rm nr}(r)$, where $p_{\rm nr}(r) = \sqrt{2\mu[E - V(r)]}$ is the nonrelativistic momentum of effective particle with the reduced mass μ.

In ultrarelativistic limit ($\chi \to \infty$) inequality in (21) has the form

$$\lambda'\left|\frac{d}{dr}\ln\left|\ln\frac{\lambda'(r)}{\lambda'}\right|\right| \ll 1,\ \lambda'(r) = \frac{\lambda'}{\sinh\chi(r)}. \tag{23}$$

We note that for the case of interaction between two equal-mass relativistic particles ($m_1 = m_2 = m$), conditions (21) and (23) coincide with conditions that were obtained in [8].

Relativistic leptonic decay widths of the mesons for the states with $\ell = 0$ in accordance with relations (1), (7), and (11) is given by

$$\Gamma(e^+e^-) = \frac{16\pi\alpha^2 e_q^2}{M_n^2} \lim_{r\to i\lambda'} \left|\psi_{q'_n}(\mathbf{r})\right|^2 \Big|_{\ell=0} = \frac{16\pi\alpha^2 e_q^2}{M_n^2} \lim_{r\to i\lambda'} \left|\frac{\varphi_0(r,\chi_n)}{r}\right|^2, \quad (24)$$

where the importance of rapidity of χ_n corresponds to the importance of energy $M = M_n = (m'^2c^2/\mu)\cosh\chi_n$ for given of level n.

We will begin considering for the case of confining potential $V_c(r)$ moreover $V_c(0) = 0$ and does not depend on energy. The JWKB approximation for the RQP radial waves function in the case of $\ell = 0$ in accordance with relations (16) and (17) is given by expression ($C_L = C_0$)

$$\varphi_0(r,\chi_n) = \frac{C_0}{\sqrt{\sinh\chi(r)}} \sin\left[\frac{1}{\lambda'}\int_{r_L}^{r} dr'\chi(r') + \frac{\pi}{4}\right], \quad (25)$$

where the value $\chi(r)$ is determined in (22), and the normalization constant C_0 is found from the normalization condition

$$\int d\mathbf{r} \left|\frac{\varphi_0(r,\chi_n)}{r}\right|^2 = 4\pi\int_0^\infty dr\, |\varphi_0(r,\chi_n)|^2 = 1. \quad (26)$$

In the region of applicability of the relativistic JWKB method the argument of sine wave in (25) is the quickly oscillating function. So the square of sine wave in (26) can be replaced, either as in the nonrelativistic case, on its the average importance equal 1/2. Instead of Eq. (26) we then obtain

$$2\pi|C_0|^2 \int_{r_L}^{r_R} \frac{dr}{\sinh\chi(r)} = 1. \quad (27)$$

Differentiation of the JWKB quantization condition (19) under $\ell = 0$ on the total energy M_n in accordance with the expressions (13), (22) and condition (18) for the turning points $r_{L,R}$, gives

$$\int_{r_L}^{r_R} \frac{dr}{\sinh\chi(r)} = \frac{\pi\lambda' m'^2 c^2}{\mu}\frac{dn}{dM_n}. \quad (28)$$

Then, from expressions (27) and (28) we find

$$|C_0|^2 = \frac{\mu}{2\pi^2\lambda' m'^2 c^2}\frac{dM_n}{dn}. \tag{29}$$

By using relations (25) and (29), and at calculations by considering only main members on λ', we come to the relativistic expression

$$\lim_{r\to i\lambda'}\left|\frac{\varphi_0(r,\chi_n)}{r}\right|^2 = \frac{\mu\sinh\chi_n}{2\pi^2\lambda'^3 m'^2 c^2}\frac{dM_n}{dn} = \frac{\sinh^2\chi_n}{2\pi^2\lambda'^3}\frac{d\chi_n}{dn}. \tag{30}$$

We note that expression (30) in the case of equal masses ($m_1 = m_2 = m$) differs from the corresponding expression in [8]

$$\lim_{r\to 0}\left|\frac{\varphi_0(r,\chi_n)}{r}\right|^2 = \frac{\chi_n^2}{4\pi^2\lambda^2\hbar c\sinh\chi_n}\frac{dM_n}{dn} = \frac{1}{6\pi^2\lambda^3}\frac{d(\chi_n)^3}{dn},\ \lambda = \frac{\hbar}{mc}$$

by the relativistic factor $(\chi_n/\sinh\chi_n)^2$ since they are calculated in points $r = i\lambda'$ and $r = 0$. The expression (30) in the case of equal masses and $\hbar = c = 1$ coincides in form with its the nonrelativistic of analogue (2) with the change $v_n^{\rm nr} \to v_n^{\rm rel}$; however, the role of velocity parameter plays the velocity of efficient relativistic particle, appearing as a two-body system. In the nonrelativistic limit $v_n^{\rm rel} \to 0$ both expressions coincide. In Fig. 1, we present the behavior of the function $R_{\rm Conf}$ defined as the ratio of the relativistic square of the wave function for the $\ell = 0$ state in (30) and (5) to the nonrelativistic expression in (2) as a function of the variable v for the case of confining potential $V_c(r)$ ($V_c(0) = 0$), $m_1 = m_2 = m$ and $\hbar = c = 1$. The solid curve corresponds to expression in (30), while the dashed curve corresponds to expression in (5). From Fig. 1, one can see that, in the nonrelativistic limit ($v \to 0$) the relativistic expressions in (30) and (5) reproduces the nonrelativistic result. The new expression (30) differs significantly from the relativistic of expression (5) presence additional multiplier $\sqrt{1-(v_n^{\rm rel})^2}$ in its denominator.

The JWKB quantization condition (19) under $\ell = 0$ with the linear potential $V_c(r) = \sigma r$ is given by expression

$$\chi_n\cosh\chi_n - \sinh\chi_n = \frac{\pi\sigma\mu\lambda'}{m'^2c^2}\left(n - \frac{1}{4}\right),\ n = 1,2,\ldots.$$

Eq. (30) takes then the form

$$\lim_{r\to i\lambda'}\left|\frac{\varphi_0(r,\chi_n)}{r}\right|^2 = \frac{\sigma\mu}{2\pi\lambda'^2 m'^2 c^2}\frac{\sinh\chi_n}{\chi_n}, \tag{31}$$

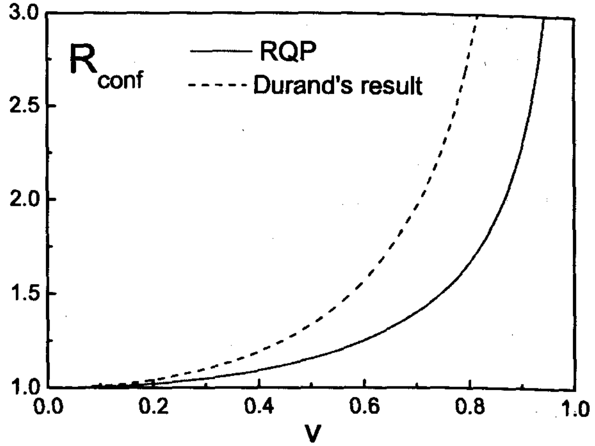

Figure 1. Behavior of the ratio $R_{\rm Conf}$ of the relativistic to the nonrelativistic square of the wave function for the $\ell = 0$ as a function of the variable v for the case of confining potential $V_c(r)$. The solid and dashed curves represent, respectively, the ratio of the expressions (30) and (5) to the nonrelativistic expression (2) in the case of $m_1 = m_2 = m$, $\hbar = c = 1$.

which differs from the corresponding of nonrelativistic expression only by the relativistic factor $\sinh\chi_n/\chi_n$, not giving contribution in the nonrelativistic limit ($\sinh\chi_n/\chi_n \to 1$ under $c \to \infty$) and serves the measure of contribution of relativistic effects [23].

Thus, the relativistic leptonic decay widths of the mesons for the states with $\ell = 0$ in the case of confining potential $V_c(r)$ ($V_c(0) = 0$) in accordance with relations (24) and (30) is given by formula

$$\Gamma(e^+e^-) = \frac{8\alpha^2 e_q^2 \mu}{\pi\lambda'^3 m'^2 c^2}\frac{\sinh\chi_n}{M_n^2}\frac{dM_n}{dn} = \frac{8\alpha^2 e_q^2}{\pi\lambda'^3}\frac{\sinh^2\chi_n}{M_n^2}\frac{d\chi_n}{dn}. \qquad (32)$$

The formula (32) for the linear potential $V_c(r) = \sigma r$ according to expression (31) takes the form

$$\Gamma(e^+e^-) = \frac{8\alpha^2 e_q^2 \sigma\mu}{\lambda'^2 m'^2 c^2}\frac{1}{M_n^2}\frac{\sinh\chi_n}{\chi_n}. \qquad (33)$$

Now we shall consider the case, when the interaction between two relativistic particles of arbitrary masses m_1 and m_2 is realized by means of quasipotential (8). In the field of such the potential it is already possible the existence of energy levels $M_n < m'^2c^2/\mu$. Importance of the energy M_n in the field of the potential (8) for given of level n can be determined from the JWKB quantization condition (19). The JWKB radial wave function in the potential (8) for given fixed of energy $M_n < m'^2c^2/\mu$ near of the begining coordinates where a Coulomb interaction dominates in the potential (8), can be then approximated for r small by the Coulomb radial wave function for which its the exact form is the known [21], [24], [25]

$$\varphi_\ell^{\rm Coul}(r, \chi_n = i\kappa_n) = C_\ell^{\rm Coul}\,(-\rho)^{(\ell+1)} \exp\left[-\rho\kappa_n + \frac{i\tilde{\alpha}_{\rm S}\kappa_n}{2\sin\kappa_n} - i(\ell+1)\kappa_n\right]$$
$$\times F\left(\ell+1-\frac{\tilde{\alpha}_{\rm S}}{2\sin\kappa_n}, \ell+1-i\rho; 2\ell+2; 1-e^{-2i\kappa_n}\right). \qquad (34)$$

Here $\rho = r/\lambda'$, $\tilde{\alpha}_{\rm S} = \dfrac{2\kappa_{\rm S}^2\mu}{\lambda' m'^2c^2}$, $\kappa_n = \arccos\dfrac{\mu M_n}{m'^2c^2}$, and the function $(-\rho)^{(l)} = i^l\Gamma(l+i\rho)/\Gamma(i\rho)$ is the generalized power [20], where $\Gamma(z)$ is gamma-function. The JWKB solution with the potential (8), taken in classically forbidden region ($r > r_{\rm R}$), for $\ell = 0$ and for fixed importance of the energy of bound state with $M_n = (m'^2c^2/\mu)\cos\kappa_n$, and for enough of the large value of ρ but such, where the Coulomb part will dominate in the potential (8) [8], according to relations (16) and (17) has the form ($C_{\rm R} = C_0$)

$$\varphi_0(r, i\kappa_n) = \frac{C_0}{2\sqrt[4]{1-X^2(r)}} \exp\left[-\frac{1}{\lambda'}\int\limits_{r_{\rm R}}^{r} dr' \arccos[X(r')] + \frac{i\pi}{4}\right]. \qquad (35)$$

Comparing the asymptotic expression of the Coulomb radial wave function in (34) under $\ell = 0$,

$$\varphi_\ell^{\rm Coul}(r, \chi_n = i\kappa_n)\Big|_{\rho\gg 1} \sim C_\ell^{\rm Coul}\times$$
$$\times\frac{\Gamma(2\ell+2)\exp\left[-\rho\kappa_n + \dfrac{\tilde{\alpha}_{\rm S}}{2\sin\kappa_n}\ln(2\rho\sin\kappa_n) + \dfrac{i\pi\tilde{\alpha}_{\rm S}}{2\sin\kappa_n}\right]}{(2\sin\kappa_n)^{\ell+1}\sqrt{2\pi}\exp\left(-\dfrac{\tilde{\alpha}_{\rm S}}{2\sin\kappa_n}\right)\left(\dfrac{\tilde{\alpha}_{\rm S}}{2\sin\kappa_n}\right)^{\dfrac{\tilde{\alpha}_{\rm S}}{2\sin\kappa_n}+\ell+\frac{1}{2}}},$$

with the asymptotic form the JWKB solution in (35),

$$\varphi_0(r, i\kappa_n)\big|_{\rho\gg 1} \sim \frac{C_0\sqrt{\pi\tilde{\alpha}_S}\exp\left[-\rho\kappa_n + \dfrac{\tilde{\alpha}_S}{2\sin\kappa_n}\ln(2\rho\sin\kappa_n)\right]}{2\sin\kappa_n\sqrt{2\pi}\exp\left(-\dfrac{\tilde{\alpha}_S}{2\sin\kappa_n}\right)\left(\dfrac{\tilde{\alpha}_S}{2\sin\kappa_n}\right)^{\dfrac{\tilde{\alpha}_S}{2\sin\kappa_n}+\dfrac{1}{2}}},$$

we find the relationship between the normalization constants

$$\left|C_0^{\mathrm{Coul}}\right|^2 = \pi\tilde{\alpha}_S\left|C_0\right|^2. \tag{36}$$

From expressions (29), (34), taken under $\ell = 0$, and (36), we obtain

$$\lim_{r\to i\lambda'}\left|\frac{\varphi_0(r, i\kappa_n)}{r}\right|^2 = \lim_{r\to i\lambda'}\left|\frac{\varphi_0^{\mathrm{Coul}}(r, i\kappa_n)}{r}\right|^2 = \frac{\tilde{\alpha}_S\mu}{2\pi\lambda'^3 m'^2c^2}\frac{dM_n}{dn}, \tag{37}$$

Thus, the relativistic leptonic decay widths of the mesons for the states with $\ell = 0$ and with mass $M_n = (m'^2c^2/\mu)\cos\kappa_n$ in the case of potential of form (8) in accordance with relations (24) and (37) is given as

$$\Gamma(e^+e^-) = \frac{8\alpha^2 e_q^2\tilde{\alpha}_S\mu}{\lambda'^3 m'^2c^2}\frac{1}{M_n^2}\frac{dM_n}{dn}. \tag{38}$$

Now we will consider in the field of potential (8) the importance of energy $M_n = (m'^2c^2/\mu)\cosh\chi_n > m'^2c^2/\mu$, importance which for given of level n also can be determined from the JWKB quantization condition (19). The JWKB radial wave function (25) in the potential (8) for fixed importance M_n of the energy of bound state and for enough of the large value of ρ but such, where the Coulomb interaction will dominate in the potential (8), can be then approximated by the Coulomb radial wave function for which its the exact form is the known [21], [26], [27]

$$\varphi_\ell^{\mathrm{Coul}}(r, \chi_n) = C_\ell^{\mathrm{Coul}}(-\rho)^{(\ell+1)}\times$$
$$\times\exp\left[i\rho\chi_n + \frac{i\tilde{\alpha}_S\chi_n}{2\sinh\chi_n} - (\ell+1)\chi_n + i\pi(\ell+1)\right]\times$$
$$\times F\left(\ell+1-\frac{i\tilde{\alpha}_S}{2\sinh\chi_n}, \ell+1-i\rho; 2\ell+2; 1-e^{-2\chi_n}\right). \tag{39}$$

From the comparing of asymptotic form of JWKB solution (25),

$$\varphi_0(r, \chi_n)\big|_{\rho\gg 1} \sim \frac{C_0}{\sqrt{\sinh\chi_n}}\sin\left[\rho\chi_n + \frac{\tilde{\alpha}_S}{2\sinh\chi_n}\ln(2\rho\sinh\chi_n)\right],$$

with the asymptotic form of Coulomb wave function (39) under $\ell = 0$,

$$\varphi_\ell^{\mathrm{Coul}}(r,\chi_n)\big|_{\rho\gg 1} \sim \frac{2C_\ell^{\mathrm{Coul}}\Gamma(2\ell+2)\exp\left[-\dfrac{\pi\tilde{\alpha}_{\mathrm{S}}}{4\sinh\chi_n}\right]}{(2\sinh\chi_n)^{\ell+1}\left|\Gamma\left(\ell+1-\dfrac{i\tilde{\alpha}_{\mathrm{S}}}{2\sinh\chi_n}\right)\right|}\sin\left[\rho\chi_n+\right.$$

$$\left.+\frac{\tilde{\alpha}_{\mathrm{S}}}{2\sinh\chi_n}\ln(2\rho\sinh\chi_n)-\frac{\pi\ell}{2}+\arg\Gamma\left(\ell+1-\frac{i\tilde{\alpha}_{\mathrm{S}}}{2\sinh\chi_n}\right)\right],$$

we find the relationship between the normalization constants:

$$\left|C_0^{\mathrm{Coul}}\right|^2 = \sinh\chi_n\exp\left[\frac{\pi\tilde{\alpha}_{\mathrm{S}}}{2\sinh\chi_n}\right]\left|\Gamma\left(1-\frac{i\tilde{\alpha}_{\mathrm{S}}}{2\sinh\chi_n}\right)\right|^2|C_0|^2. \tag{40}$$

From expressions (29), (39) for $\ell = 0$, and (40), we obtain

$$\lim_{r\to i\lambda'}\left|\frac{\varphi_0(r,\chi_n)}{r}\right|^2 = \lim_{r\to i\lambda'}\left|\frac{\varphi_0^{\mathrm{Coul}}(r,\chi_n)}{r}\right|^2 =$$

$$= \frac{\mu\sinh\chi_n}{2\pi^2\lambda'^3m'^2c^2}S_{\mathrm{RQP}}\left(u_n^{\prime\mathrm{rel}}\right)\frac{dM_n}{dn} = \frac{\mu^2u_n^{\prime\mathrm{rel}}}{2\pi^2\lambda'^3m'^3c^2}S_{\mathrm{RQP}}\left(u_n^{\prime\mathrm{rel}}\right)\frac{dM_n}{dn}, \tag{41}$$

where

$$S_{\mathrm{RQP}}(u_n^{\prime\mathrm{rel}}) = \frac{X_{\mathrm{RQP}}(u_n^{\prime\mathrm{rel}})}{1-\exp\left[-X_{\mathrm{RQP}}(u_n^{\prime\mathrm{rel}})\right]},\ X_{\mathrm{RQP}}(u_n^{\prime\mathrm{rel}}) = \frac{2\pi\alpha_{\mathrm{S}}}{u_n^{\prime\mathrm{rel}}},\ \alpha_{\mathrm{S}} = \frac{\kappa_{\mathrm{S}}^2}{\hbar c} \tag{42}$$

is the relativistic Coulomb S-factor [27], [28], the value

$$u_n^{\prime\mathrm{rel}} = \frac{2u_n}{\sqrt{1-u_n^2}} \tag{43}$$

is the relative velocity of an effective relativistic particle with mass m' emerging instead of the system of two particles, and the velocity u_n is

$$u_n = \sqrt{1-\frac{4m'^2c^4}{M_n^2-(m_1-m_2)^2c^4}}. \tag{44}$$

The relativistic leptonic decay widths of the mesons for the states with $\ell = 0$ and with mass $M_n = (m'^2c^2/\mu)\cosh\chi_n$ in the case of potential of form (8) in accordance with relations (24) and (41) is given then by

$$\Gamma(e^+e^-) = \frac{8\alpha^2e_q^2\mu^2u_n^{\prime\mathrm{rel}}}{\pi\lambda'^3m'^3c^2}\frac{S_{\mathrm{RQP}}\left(u_n^{\prime\mathrm{rel}}\right)}{M_n^2}\frac{dM_n}{dn}. \tag{45}$$

Eq. (45) under $M_n < m'^2c^2/\mu$ continuously moves over to equality (38).

We note that in the case of $m_1 = m_2 = m, \hbar = c = 1$ the relativistic formula in (41) coincides in form with its the nonrelativistic of analogue (3) with the change $2v_n^{\rm nr} \to u_n'^{\rm rel}$ and $4\alpha_{\rm S}/3 \to \kappa_{\rm S}^2$. In the nonrelativistic limit $v_n^{\rm rel} \to 0$ both expressions coincide. In Fig. 2, we present the behavior of

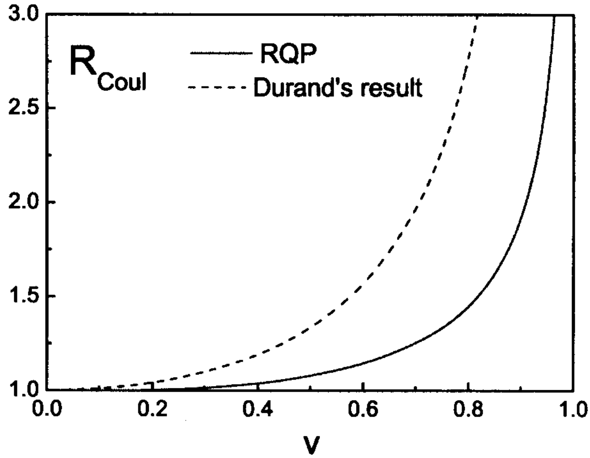

Figure 2. Behavior of the ratio $R_{\rm Coul}$ of the relativistic to the nonrelativistic square of the wave function for the $\ell = 0$ as a function of the variable v for the case of potential (8). The solid and dashed curves represent, respectively, the ratio of the expressions (41) and (6) to the nonrelativistic expression (3) in the case of $m_1 = m_2 = m, \hbar = c = 1$.

the function $R_{\rm Coul}$ defined as the ratio of the relativistic square of the wave function for the $\ell = 0$ state in (41) and (6) to the nonrelativistic expression in (3) as a function of the variable v for the case of potential (8), $m_1 = m_2 = m$ and $\hbar = c = 1$. The solid curve represents the ratio of the expression given by (41) to the nonrelativistic expression in (3), while the dashed curve stands for the ratio of the expression in (6) to the nonrelativistic expression in (3). From Fig. 2, one can see that, in the nonrelativistic limit ($v \to 0$) the relativistic expressions in (41) and (6) reproduces the nonrelativistic result. The new expression (41) differs significantly from the relativistic of expression (6) presence additional multiplier $\sqrt{1-(v_n^{\rm rel})^2}$ in its denominator. This is connected with formula (26) in [29] that was obtained in the nonrelativistic case. This has not allowed the author in [29] in the relativistic case it is correct to define the normalization factor of wave function. In [29] this factor was chosen equal unit, that lead to difference in the relativistic limit ($v_n^{\rm rel} \to 1$) in formulas (5) and (6) because of factor $\sqrt{1-(v_n^{\rm rel})^2}$ in their denominators.

5 Conclusion

In the present study, the new relativistic expressions for the leptonic decay widths of the mesons for the states with $\ell = 0$ in the case of potential of form (8) were obtained. The consideration was done within the framework of completely covariant of the RQP approach in QFT, formulated in the relativistic configurational representation in the case of two particles of arbitrary masses. For this aim the finite-difference quasipotential equation in form of Schrödinger in the relativistic configuration representation for the interaction of two relativistic particles of arbitrary masses [10] was solved by the relativistic JWKB method. The JWKB quantization condition and the condition of applicability of the relativistic JWKB method in the case of two particles of arbitrary masses were obtained.

The new relativistic expressions (30) and (41) in the case of equal masses coincide in form with their the nonrelativistic of analogues (2) and (3) with the change $2v_n^{\rm nr} \to u_n^{\prime\rm rel}$ and $4\alpha_{\rm S}/3 \to \kappa_{\rm S}^2$. Consequently, the role of velocity parameter plays the velocity of efficient relativistic particle, appearing as a two-body system. In the nonrelativistic limit $u_n^{\prime\rm rel} \to 0$ both expressions coincide. The new expression (41) differs significantly from the relativistic of expression (6) presence additional multiplier $\sqrt{1-(v_n^{\rm rel})^2}$ in its denominator because of incorrect determined of the normalization factor to wave function.

As the new relativistic expressions for the leptonic decay widths of the mesons were obtained within the framework of completely covariant method, one can expect that these leptonic widths takes into account more adequately relativistic nature of interaction.

Acknowledgments

The authors would like to thank Yu. S. Vernov, E. A. Kuraev, and R. N. Faustov for interest in this work and valuable discussions. This work was partly supported by the BelRFFR-JINR grant F10D-001 and the Belarus State Program of Basic Research "Convergence".

References

[1] B. Durand and L. Durand, *Phys. Rev.* D, **30**, 1904 (1984).

[2] V. A. Matveev, B. V. Struminsky, A. N. Tavkhelidze, Preprint JINR P-2524, Dubna, 1965.

[3] W. F. Weisskopf, R. Van Royen, *Nuovo Cim.*, **50**, 617 (1967); **51**, 583 (1967).

[4] H. I. Schnitzer, *Phys. Rev.* D, **13**, 74 (1976).

[5] I. S. Kang, H. I. Schnitzer, *Phys. Rev.* D, **12**, 841 (1975); 2791 (1975).

[6] R. Barbieri, R. Gatto, R. Kögerler and Z. Kunszt, *Phys. Lett.* B, **57**, 455 (1975).

[7] E. Etim and L. Schülke , *Nuovo Cim.* A, **77**, 347 (1983).

[8] N. B. Skachkov, I. L. Solovtsov, *Yad. Fiz.*, **31**, 1332 (1980).

[9] A. A. Logunov and A. N. Tavkhelidze, *Nuovo Cim.*, **29**, 380 (1963).

[10] V. G. Kadyshevsky, M. D. Mateev, and R. M. Mir-Kasimov, *Yad. Fiz.*, **11**, 692 (1970).

[11] C. Quigg and J. Rosner, *Phys. Rev.* D, **17**, 2364 (1978).

[12] J. S. Bell and J. Pasupathy, *Phys. Lett.* B, **83**, 389 (1979).

[13] J. Pasupathy and V. Sing *Z. Phys.* C, **10**, 23 (1981).

[14] J. S. Bell and J. Pasupathy, *Z. Phys.* C, **2**, 183 (1979).

[15] N. Fröman and P. O. Fröman, *J. Phys.* (Paris), **42**, 1491 (1981).

[16] B. Durand and L. Durand, *Phys. Rev.* D, **25**, 2312 (1982).

[17] B. Durand and L. Durand, *Phys. Lett.* B, **113**, 338 (1982).

[18] E. A. Tainov, *Phys. Lett.* B, **97**, 283 (1980); *Z. Phys.* C, **10**, 87 (1981).

[19] V. G. Kadyshevsky, *Nucl. Phys. B*, **6**, 125 (1968).

[20] V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, *Nuovo Cim. A*, **55**, 233 (1968).

[21] V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, *Sov. J. Purt. Nucl.*, **2**, 69 (1972).

[22] V. N. Kapshay, V. I. Savrin, N. B. Skachkov, *Theor. Math. Phys.*, **69**, 400 (1986).

[23] N. B. Skachkov, *Theor. Math. Phys.*, **25**, 313 (1975).

[24] V. G. Kadyshevsky, R. M. Mir-Kasimov, and M. Friman, *Yad. Fiz.*, **9**, 646 (1969).

[25] M. Freeman, M. D. Mateev, R. M. Mir-Kasimov, *Nucl. Phys. B*, **12**, 197 (1969).

[26] Yu. D. Chernichenko, *Vestsi Nats. Akad. Navuk Belarusi, Ser. fiz.-mat. navuk*, **4**, 81 (2009).

[27] O. P. Solovtsova, Yu. D. Chernichenko, *Theor. Math. Phys.*, **166**, 225 (2011).

[28] O. P. Solovtsova, Yu. D. Chernichenko, *Yad. Fiz.*, **73**, 1 (2010).

[29] B. Durand and L. Durand, *Phys. Rev.* D, **28**, 396 (1983).