Reconstructing a non-local separable interaction in the framework of the relativistic quasipotential model

Yu.D.Chernichenko

Department of Mathematics, Pavel Sukhoy Gomel State Technical University,
October Avenue 48, 246746 Gomel, Belarus,
Fax: + 375–232–479165; Tel: +375–232–478487

In the framework of the relativistic quasipotential approach to the quantum field theory, a method is developed according to which a non-local separable quasipotential describing the interaction between two relativistic spinless particles of unequal masses can be reconstructed by of the phase shift and bound-state energies.

It was proven by Gelfand and Levitan [1, 2], Marchenko [3], and Krein [4, 5] that the inverse problem can in principle be solved in the framrwork of nonrelativistic theory. They obtained the linear integral equations in two versions, which served as a basis for a further development of inverse-problem theory. The most-complete survey of this theory was given in the monographs of Chadan and Sabatier [6] and Zakhariev and Suzko [7].

In the most of studies, however, the problem of reconstructing interaction is formulated on the basis of the non-relativistic Schrödinger equation. Therefore, the problem of reconstructing interaction for essentially relativistic systems – in particular, within the relativistic quasipotential approach [8] – is yet remained important.

Within the relativistic quasipotential approach proposed in [9], the problem is considered here for the case where a non-local separable quasipotential simulating the interaction between two relativistic spinless particles of unequal masses ($m_1 \neq m_2$) must be reconstructed on the basis of the phase shift and bound-state energies. The given approach is based on the expression that was found by the present author for the phase shift and which has the form [10] (we use the system of units where $\hbar = c = 1$)

$$\mathrm{tg}\delta_l(\chi') = -\frac{\pi}{2}\mathrm{sh}^{-1}\chi' A_l(\chi')\left[1 + \mathrm{P}\frac{1}{2}\int\limits_0^\infty d\chi \frac{A_l(\chi)}{\mathrm{ch}\chi - \mathrm{ch}\chi'}\right]^{-1}, \tag{1}$$

where the quantity χ' is defined via the relation $E_{q'} = m'\sqrt{1 + (\vec{q'}/m')^2} = m'\mathrm{ch}\chi'$, $m' = \sqrt{m_1 m_2}$, and

$$A_l(\chi') = \frac{2}{\pi}\varepsilon_l Q_l^2(\mathrm{cth}\chi')\left|\tilde{V}_l(\chi')\right|^2, \quad \varepsilon_l = \pm 1. \tag{2}$$

Here, $Q_l(z)$ is a Legendre function of the second kind.

In order to find the quasipotential $V_l(r)$ on the basis of the phase shift $\delta_l(\chi')$, it is necessary to solve the integral equation (1) concerning of the function $A_l(\chi')$. After that, the function $\tilde{V}_l(\chi')$ is determined from Eq.(2). The quasipotential $V_l(r)$ is then reconstructed by performing the relativistic Hankel transformation

$$V_l(r) = \frac{2}{\pi}\int\limits_0^\infty d\chi Q_l(\mathrm{cth}\chi)\tilde{V}_l(\chi)S_l(\chi, r). \tag{3}$$

Here, the function $S_l(\chi, r)$ is a free solution of finite-difference quasipotential equation in configuration space [11].

In particular, the relativistic Hankel transformation (3) at $l = 0$ reduces to the conventional Fourier transformation

$$V_0(r) = \frac{2}{\pi}\int\limits_0^\infty d\chi \chi \tilde{V}_0(\chi)\sin r\chi.$$

We assume that the phase shift $\delta_l(\chi')$ in Eq.(1) is a function continuous in the sence of Hölder with a positive index and that, for $\chi' \to \infty$, it bahaves as

$$\delta_l(\chi') = O\left[(\chi')^{-\gamma}\right], \quad l \geq 0, \quad \gamma > 1. \tag{4}$$

These constraints are necessary and sufficient for the quasipotential to satisfy the condition

$$rV_l(r) \in L_1(0, \infty), \tag{5}$$

which ensures the uniqueness of the inverse-problem solution. We therefore assume that, as χ' increases, the phase shift $\delta_l(\chi')$ intersects the straight lines $\delta_l(\chi') = \pi n$ $(n = 0, 1, 2, ...)$ from above.

Suppose that there exist ν_l ($l \geq 0$) scattering states at energies satisfying the conditions

$$E'_{Rn} = m'\mathrm{ch}\chi'_{Rn} \geq m', \quad n = 0, 1, ..., \nu_l - 1. \tag{6}$$

We then have

$$\delta_l(0) = \pi\nu_l. \tag{7}$$

In this case $\varepsilon_l = +1$, while the scattering states energies $E'_{Rn} \geq m'$ are found by relation

$$\delta_l(\chi'_{Rn}) = \pi n, \quad n = 0, 1, 2, ..., \nu_l - 1. \tag{8}$$

The integral equation (1) can be reduced to the form

$$A_l(\mathrm{arch}x)g_l^{-1}(x) = 1 + \frac{1}{\pi}\mathrm{P}\int\limits_1^\infty dt\frac{\Psi_l(t)h_l^*(t)}{t-x}, \tag{9}$$

where $x = \mathrm{ch}\chi'$ and where we introduced the following notation:

$$\begin{aligned}
\Psi_l(x) &= A_l(\mathrm{arch}x)g_l^{-1}(x)\left[1 + (i\pi/2)g_l(x)(x^2-1)^{-1/2}\right], \qquad (10)\\
g_l(x) &= -(2/\pi)(x^2-1)^{1/2}\mathrm{tg}\Delta_l(x),\\
\Delta_l(x) &= \delta_l(\mathrm{arch}x),\\
h_l(x) &= (\pi/2)g_l(x)(x^2-1)^{-1/2}\left[1 - (i\pi/2)g_l(x)(x^2-1)^{-1/2}\right]^{-1} =\\
&= -\sin\Delta_l(x)\exp\left[-i\Delta_l(x)\right].
\end{aligned}$$

With the aid of the representation

$$1/(\alpha - io) = i\pi\delta(\alpha) + \mathrm{P}(1/\alpha),$$

Eq.(9) can be recast into the form

$$\Psi_l(x) = 1 + \frac{1}{\pi}\int\limits_1^\infty dt\frac{\Psi_l(t)h_l^*(t)}{t-x-io}. \tag{11}$$

If the function $\Psi_l(x)$ is continuos in the sence of Hölder and if the integral in Eq.(11) converges then the function

$$H_l(z) = 1 + \frac{1}{\pi}\int\limits_1^\infty dt\frac{\Psi_l(t)h_l^*(t)}{t-z} \tag{12}$$

is analytic in the complex plane of the variable z with the cut from 1 to $+\infty$, and besides the relation

$$\lim_{|z|\to\infty} H_l(z) = 1 \tag{13}$$

holds in all directions. Hence, a solution of the integral Eq.(11) can be represented as

$$\Psi_l(x) = H_l(x_+) = \lim_{\eta\to+0} H_l(x+i\eta), \quad 1 \le x \le \infty. \tag{14}$$

By substituting the solution in (14) into the expression for the discontinuity suffered by the function $H_l(z)$ upon traversing the cut, that is

$$H_l(x_+) - H_l(x_-) = 2i\Psi_l(x)h_l^*(x) = -2i\sin\Delta_l(x)\exp\left(i\Delta_l(x)\right)\Psi_l(x), \tag{15}$$

we arrive at the homogeneous Riemann–Hilbert equation for the function $H_l(z)$:

$$H_l(x_+)\exp\left(2i\Delta_l(x)\right) - H_l(x_-) = 0, \quad 1 \le x \le \infty. \tag{16}$$

A particular solution satisfying Eq.(16) and the condition in (13) has the form

$$\tilde{H}_l(z) = \exp\left[\omega_l(z)\right], \tag{17}$$

where

$$\omega_l(z) = -\frac{1}{\pi}\int_1^\infty dt\frac{\Delta_l(t)}{t-z}. \tag{18}$$

Besides

$$\lim_{|z|\to\infty} \omega_l(z) = 0, \tag{19}$$

and

$$\omega_l(z) \sim \frac{1}{\pi}\Delta_l(1)\ln|1-z| \quad \text{for} \quad z\to 1, \tag{20}$$

which holds in all directions, as follows from the assumptions on the behavior of the phase shift and from the conditions in (4) and (7). Therefore, the function $\tilde{H}_l(z)$ has a zero of order ν_l at the point $z=1$.

Thus, according to (14), (17), and (18), the partucular solution to the nonhomogeneous integral equation (11) has the form

$$\tilde{\Psi}_l(x) = \exp\left[\alpha_l(x) - i\Delta_l(x)\right], \tag{21}$$

where

$$\alpha_l(x) = -\frac{1}{\pi}\mathrm{P}\int\limits_1^\infty dt\frac{\Delta_l(t)}{t-x}. \tag{22}$$

It should be noted that the function given by (21) is regular at $x = 1$ (it has a zero of order ν_l at this point), is continuous in the sense of Hölder with the same index as the phase shift, and is limited for $x \to +\infty$. All this is consistent with the a priori assumptions on its properties.

A general solution to the homogeneous equation

$$\Psi_{lo}(x) = \frac{1}{\pi}\int\limits_1^\infty dt\frac{\Psi_{lo}(t)h_l^*(t)}{t-x-io} \tag{23}$$

has the form (14), as before, while the function

$$H_{lo}(z) = \frac{1}{\pi}\int\limits_1^\infty dt\frac{\Psi_{lo}(t)h_l^*(t)}{t-z} \tag{24}$$

is analitic in the complex plane of the variable z with the cut from 1 to $+\infty$, and besides the relation

$$\lim_{|z|\to\infty} H_{lo}(z) = 0 \tag{25}$$

holds in all directions. Finally, this function satisfies the homegeneous Riemann–Hilbert equation (16). A general solution to this equation will be sought in the form

$$H_{lo}(z) = \sum_{k=1}^{m} A_{k-1}\frac{\exp\left[\omega_l(z)\right]}{(z-1)^k}. \tag{26}$$

Substituting (26) into (16) and requiring that the function $H_{lo}(z)$ be finite at $z = 1$, we obtain $m = \nu_l$. Hence, we have

$$\Psi_{lo}(x) = H_{lo}(x_+) = \sum_{k=1}^{\nu_l} A_{k-1}\frac{\exp\left[\alpha_l(x) - i\Delta_l(x)\right]}{(x-1)^k}. \tag{27}$$

It is obvious that, as in the case of a particular solution, the function in (27) satisfies Eq.(16) and possesses all the required properties.

Therefore, by using the notation in (10) and transforming the sum as a product, we can recast the general solution to the integral equation (11) into the form

$$A_l(\chi') = -\frac{2}{\pi}\text{sh}\chi' \sin\delta_l(\chi') \exp\left[\alpha_l(\text{ch}\chi')\right] \times \prod_{n=0}^{\nu_l-1}\left[1-\frac{\text{ch}\chi'_{Rn}-1}{\text{ch}\chi'-1}\right], \quad (28)$$

where

$$\alpha_l\left(\text{ch}\chi'\right) = -\frac{1}{\pi}\text{P}\int_0^{\infty} d\chi \frac{\text{sh}\chi\delta_l(\chi)}{\text{ch}\chi - \text{ch}\chi'}. \quad (29)$$

We note that, in accordance with the definition in (2), the function $A_l(\chi')$ is of fixed sign at all values of χ', and so far as $\varepsilon_l = +1$, that it must be positive.

Thus, the solution in (28) is completely determined by the phase shift so far as χ'_{Rn} is also determined by its the behaviour. Moreover, it follows from expressions (28) and (29) that the function $A_l(\chi')$ is continuous in the sense of Hölder and that, for $\chi' \to +\infty$, it behaves as

$$\text{O}\left[e^{\chi'}(\chi')^{-\gamma}\right], \quad \gamma > 1, \quad (30)$$

provided that the phase shift satisfies condition (4).
This in turn implies that the quasipotential $V_l(r)$ satisfies condition (5).

The case where $\varepsilon_l = -1$ and where there are ν_l the scattering states at energies satisfying the conditions (6), and n_l the bound states whose energies lie the in the range

$$0 \le E'_{ik} = m'\cos\kappa'_{ik} < m', \quad \chi'_{ik} = i\kappa'_{ik}, k = 0, 1, ..., n_l - 1, \quad (31)$$

is considered in the same way.

Besides, by the Levinson theorem, we have

$$\delta_l(0) = \pi(\nu_l + n_l). \quad (32)$$

In accordance with expression (20), the function $\tilde{H}_l(z)$ therefore has a zero of order $(\nu_l + n_l)$ at $z = 1$. Further following in the same way as for the case of $\varepsilon_l = +1$ and considering that the function $A_l(\chi')$ must now retain a minus sign at all values of χ', so far as $\varepsilon_l = -1$, we obtain

$$\begin{aligned} A_l(\chi') &= -\frac{2}{\pi}\text{sh}\chi' \sin\delta_l(\chi') \exp\left[\alpha_l(\text{ch}\chi')\right] \prod_{n=0}^{\nu_l-1}\left[1-\frac{\text{ch}\chi'_{Rn}-1}{\text{ch}\chi'-1}\right] \times \quad (33) \\ &\times \prod_{k=0}^{n_l-1}\left[1+\frac{1-\cos\kappa'_{ik}}{\text{ch}\chi'-1}\right]. \end{aligned}$$

Thus, the function $A_l(\chi')$ is completely determined by the phase shift and bound states too, and its sign is contrary to the sign of the phase shift for $\chi' \to +\infty$.

In order to reconstruct the quasipotential $V_l(r)$ by means of the transformation in (3), we can introduce the function

$$\hat{V}_l(\chi') = \prod_{k=0}^{n_l-1}\left[\frac{\mathrm{sh}(\chi'/2) + i\sin(\kappa'_{ik}/2)}{\mathrm{sh}(\chi'/2) - i\sin(\kappa'_{ik}/2)}\right] Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')\Big/A_l^{as}(\chi'), \tag{34}$$

where $A_l^{as}(\chi')$ is asymptotic form of the function

$$\left|Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')\right| = \sqrt{(\pi/2)\varepsilon_l A_l(\chi')}$$

for $|\chi'| \to \infty$.

The function $\hat{V}_l(\chi')$ is analitic in the region $\mathrm{Im}\chi' > 0$, it is continuous for $\mathrm{Im}\chi' \geq 0$ and satisfies the condition

$$\hat{V}_l(\chi') = 1 + o(1), \quad |\chi'| \to \infty, \tag{35}$$

provided that the condition in (5) is carried out. Besides, the function $\hat{V}_l(\chi')$ vanishes nowhere for $\mathrm{Im}\chi' > 0$. Hence, the function $\ln \hat{V}_l(\chi')$ is analitic in the region $\mathrm{Im}\chi' > 0$ and tends to zero at infinity because of the estimate in (35). Therefore, we can apply the integral Hilbert transformation to the real and the imaginary parts of the function $\ln \hat{V}_l(\chi')$, setting

$$Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi') = \left|Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')\right| \exp\left[i\Phi_l(\chi')\right]. \tag{36}$$

We then obtain

$$\mathrm{Im}\ln\hat{V}_l(\chi') = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} d\mathrm{sh}(\chi/2)\frac{\mathrm{Re}\ln\hat{V}_l(\chi)}{\mathrm{sh}(\chi/2) - \mathrm{sh}(\chi'/2)} = \tag{37}$$

$$= i\ln\left[\left|Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')\right|/A_l^{as}(\chi')\right] - \frac{1}{\pi}\int_{-\infty}^{\infty} d\mathrm{sh}(\chi/2)\frac{\ln\left[\left|Q_l(\mathrm{cth}\chi)\tilde{V}_l(\chi)\right|/A_l^{as}(\chi)\right]}{\mathrm{sh}(\chi/2) - \mathrm{sh}(\chi'/2) - io}$$

Combining (37) with the expression for

$$\mathrm{Re}\ln\hat{V}_l(\chi') = \ln\left[\left|Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi')\right|/A_l^{as}(\chi')\right], \tag{38}$$

we now obtain the formula

$$\ln\hat{V}_l(\chi') = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mathrm{sh}(\chi/2)\frac{\ln\left[(\pi/2)\varepsilon_l A_l(\chi)/\left(A_l^{as}(\chi)\right)^2\right]}{\mathrm{sh}(\chi/2) - \mathrm{sh}(\chi'/2)}, \tag{39}$$

which is valid in the region $\mathrm{Im}\chi' > 0$. At last, from expressions (34) and (39), it follows that

$$Q_l(\mathrm{cth}\chi')\tilde{V}_l(\chi') = A_l^{as}(\chi') \prod_{k=0}^{n_l-1} \left[\frac{\mathrm{sh}(\chi'/2) - i\sin(\kappa'_{ik}/2)}{\mathrm{sh}(\chi'/2) + i\sin(\kappa'_{ik}/2)}\right] \times \tag{40}$$
$$\times \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mathrm{sh}(\chi/2)\frac{\ln\left[(\pi/2)\varepsilon_l A_l(\chi)/\left(A_l^{as}(\chi)\right)^2\right]}{\mathrm{sh}(\chi/2) - \mathrm{sh}(\chi'/2)}\right\},$$

which is valid for $\mathrm{Im}\chi' > 0$.

Thus, a solution to the inverse problem exists and completely determined as the function $A_l(\chi')$ is found by the phase shift and bound-state energies for $l \geq 0$.

To summarize, we note that the method proposed here to reconstruct a non-local separable quasipotential simulating the interaction between two relativistic spinless particles of unequal masses actually reduces to a one-body problem. This is thanks to the possibility of representing, within the relativistic quasipotential approach to quantum field theory, the total c.m. energy of two relativistic particles of unequal masses as an expression proportional to the energy of an effective relativistic particle of mass m'.

Acknowledgments

I am grateful to Yu.S.Vernov, V.I.Savrin, and A.M.Shirokov for a permanent interest in this study and for enlightening discussions on the results presented here.

References

[1] I.M.Gel'fand and B.M.Levitan, Dokl.Akad.Nauk SSSR **77**, 557 (1951).

[2] I.M.Gel'fand and B.M.Levitan, Izv.Akad.Nauk SSSR, Ser.Mat. **15**, 309 (1951).

[3] V.A.Marchenko, Dokl.Akad.Nauk SSSR **104**, 695 (1955).

[4] M.G. Kreĭn, Dokl.Akad.Nauk SSSR **76**, 21 (1951).

[5] M.G. Kreĭn, Dokl.Akad.Nauk SSSR **76**, 345 (1951).

[6] K.Chadan and P.C.Sabatier. Inverse Problems in Quantum Scattering Theory (Springer-Verlag, New York, 1977; Mir, Moscow 1980).

[7] B.N.Zakhariev and A.A.Suzko, Direct and Inverse Problems: Potentials in Quantum Scattering (Energoatomizdat, Moscow, 1985; Springer-Verlag, Berlin, 1990).

[8] A.A.Logunov and A.N.Tavkhelidze, Nuovo Cimento **29**, 380 (1963).

[9] V.G.Kadyshevsky, Nucl. Phys. B **6**, 125 (1968).

[10] Yu.D.Chernichenko, Preprint No.88-27/48, NIIYaF MGU (Institute of Nuclear Physics, Moscow State University, Moscow, 1988).

[11] V.G.Kadyshevsky, R.M.Mir-Kasimov, and N.B.Skachkov, Yad.Fiz. **9**, 462 (1969) [Sov. J. Nucl.Phys. **9**, 265 (1969)].