Renormalization Scheme Dependence in Variational Approach to QCD

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Abstract

We present a detailed investigation of the renormalization scheme dependence for the Adler D-function in the framework of the variational approach to QCD.

Introduction. The main object in a description of the hadronic contribution of many physical processes is the hadronic correlator or the corresponding Adler D-function. Physical quantities are independent of the particular choosing a concrete renormalization scheme (RS) [1]. However, in real calculations this dependence is appeared due to a truncation of perturbative series and RS-dependence is a source of essential theoretical uncertainties which become to be large especially at low energy scale. There are no fundamental principles upon which one can choose one or another preferable RS and it is necessary to consider the stability results obtained with respect to the choose of RS.

Various methods are proposed to eliminate the effect of choosing a concrete RS in calculations of physical quantities. Here, to investigate the RS-dependence, we use the nonperturbative method of constructing the so-called floating or variational series in quantum chromodynamics proposed in [2, 3]. This approach to QCD is based on the idea of variational perturbation theory (VPT) [4, 5, 6, 7]. The expansion parameter is a new small parameter connected with the initial coupling constant by some equation. This method maintains the high energy physics and in nonperturbative region the expansion parameter remains small value. We show that the scheme dependence in the VPT is considerably less than in the standard perturbative approach (PT), and applying the VPT allows us to considerably reduce the RS-dependence.

We note here that in the framework of the APT, the RS-dependence of the Adler D-function in QCD has been discussed in detail in [8, 9, 10].

Variational perturbative theory. Analysis of the structure of the variational perturbation series shows that it can be organized in powers of the new small expansion parameter a associated with the initial coupling constant λ by following equation

$$\lambda = \frac{g^2}{(4\pi)^2} = \frac{\alpha_s}{4\pi} = \frac{1}{C}\frac{a^2}{(1-a)^3}\,. \tag{1}$$

The expansion parameter a obeys the inequality $0 \le a < 1$. The positive constant C plays the role of a variational parameter and does not depend on experimental data. The original quantity which is approximated by this expansion does not depend on the auxiliary parameter C; however, any finite approximation depends on it on account of the truncation of the series.

The variational parameter C can be defined, if one takes into account the Källén–Lehmann analyticity [1] (as it was done in the analytic perturbation theory [11]). We note also that analytic properties of the running coupling are important from the point of view phenomenological applications.

It has been demonstrated in [12, 13], a value of parameter C changes from order to order, in accordance with the phenomenon of induced convergence. It has been observed empirically in [14, 15] that the results seem to converge if the variational parameter is chosen, in each order, according to some variational principle. This induced-convergence mechanism is also discussed in [16].

The running expansion parameter $a(Q^2)$ as a function of Q^2 is determined from the renormalization group equation with an accuracy $O(a^i)$:

$$Q^2 = Q_0^2 \exp\left[\frac{C_i}{2\beta_0}\left(f_i(a) - f_i(a_0)\right)\right], \tag{2}$$

where

$$f_i(a) = \frac{2\beta_0}{C_i}\int^{\lambda} \frac{d\lambda}{\beta_i(\lambda)}\,. \tag{3}$$

In the VPT approach, the three-loop β-function as a function of the parameter a has the form

$$\beta_{VPT}(x) = -\frac{1}{C^2}\frac{2\beta_0 a^4}{(2+a)(1-a)^2}(1 + k_1 a + k_2 a^2 + k_3 a^3 + k_4 a^4 + k_5 a^5)\,, \tag{4}$$

where

$$k_1 = \frac{9}{2}, \qquad k_2 = 12 + \frac{\beta_1}{\beta_0 C}, \qquad k_3 = 25 + \frac{15}{2}\frac{\beta_1}{\beta_0 C}, \tag{5}$$
$$k_4 = 45 + \frac{63}{2}\frac{\beta_1}{\beta_0 C} + \frac{\beta_2}{\beta_0 C^2}, \qquad k_5 = \frac{147}{2} + 98\frac{\beta_1}{\beta_0 C} + \frac{21}{2}\frac{\beta_2}{\beta_0 C^2},$$

and

$$\beta_0 = 11 - \frac{2}{3}f, \quad \beta_1 = 102 - \frac{38}{3}f, \quad \beta_2^{\overline{MS}} = \frac{2857}{2} - \frac{5033}{18}f + \frac{325}{54}f^2 \tag{6}$$

are standard coefficients of perturbation theory [17], and $a = a(\lambda)$ is according to (1). Note that the three-loop β-function coefficient β_2 depends from the choosing of renormalization scheme.

The running parameter $a(Q^2)$ is defined as an implicit function of $Q^2(a)$ via the equation (2) with $f_i(a)$ in the order $O(a^i)$:

$$f_2(a) = \frac{2}{a^2} + \frac{12}{a} - \frac{9}{1-a} + 21\ln(1-a) - 21\ln a,$$

$$f_3(a) = \frac{2}{a^2} - \frac{6}{a} - \frac{18}{11}\frac{1}{1-a} - 48\ln a + \frac{624}{121}\ln(1-a) + \frac{5184}{121}\ln\left(1 + \frac{9a}{2}\right),$$

$$\cdots$$

$$f_i(a) = \frac{2}{a^2} - \frac{6}{a} - \frac{A_3}{1-a} + A_1\ln a + A_2\ln(1-a) + \sum_{k=1}^{i-2} B_k \ln(a - a_k), \tag{7}$$

where the coefficients A and B in Eq. (7) are changing from order to order.

By examination a complex plane of running expansion parameter a as a function of Q^2 from Eq. (2) and the branches of the many-valued function $a = a(Q^2)$, we required of certain analytic properties of the parameter a. This allowed us to define the values of the variational parameters C_i (see [12, 13] for details). taking into account the nonperturbative β-function and a correlation between the coefficients A and B in Eq. (7):

$$A_1 + A_2 + \sum_{k=1}^{i-2} B_k = 0, \tag{8}$$

we obtain for $f = 3$ the variational parameters C_i in the order $O(a^i)$:

$$C_3 = 3.5, \quad C_4 = 9.2, \quad C_5 = 19.1, \quad C_6 = 34.1, \quad C_7 = 55.6. \tag{9}$$

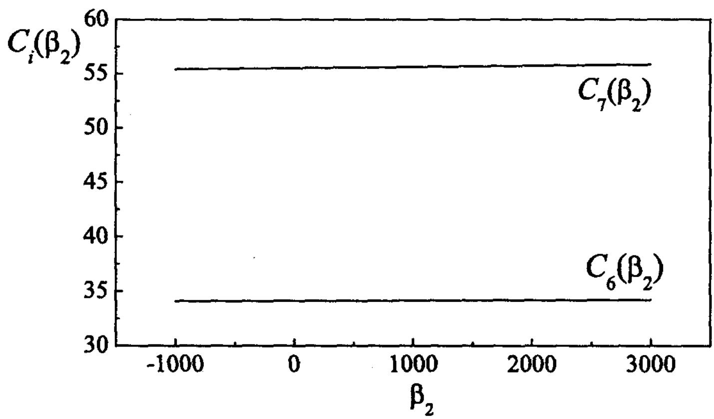

Figure 1: *Stability of the variational parameters C_6 and C_7 with respect to the RS-dependent β-function coefficient β_2.*

These values of parameters agree well with values which have been found earlier from the meson spectroscopy [2, 3].

It is important to note, that the value of the parameter C is not too sensitive to the change of the RS-dependent three-loop β-function coefficient β_2 (see Fig 1). We have found for the coefficient C_7 in the different renormalization schemes ($\overline{\text{MS}}$, K, H and V)[1] with the scheme dependent three-loop β-function coefficient β_2:

$$\beta_2(\overline{\text{MS}}) = 643.83, \quad \beta_2(K) = -1644.04, \quad \beta_2(H) = 0, \quad \beta_2(V) = 1766.88,$$

where

$$C_{\overline{\text{MS}}} = 55.5901, \quad C_{\text{V}} = 55.7051, \quad C_{\text{H}} = 55.5238, \quad C_{\text{K}} = 55.3531\,.$$

In order to illustrate the renormalization scheme dependence problem let us consider the three-loop perturbative β-function and the VPT β-function (4) in the different renormalization schemes. The calculations were performed in the schemes $\overline{\text{MS}}$, K, H and V. The behavior of the β-function in the PT

[1]We use the K-scheme, which is interesting in that there is a fixed point for the three-loop running coupling point [18].

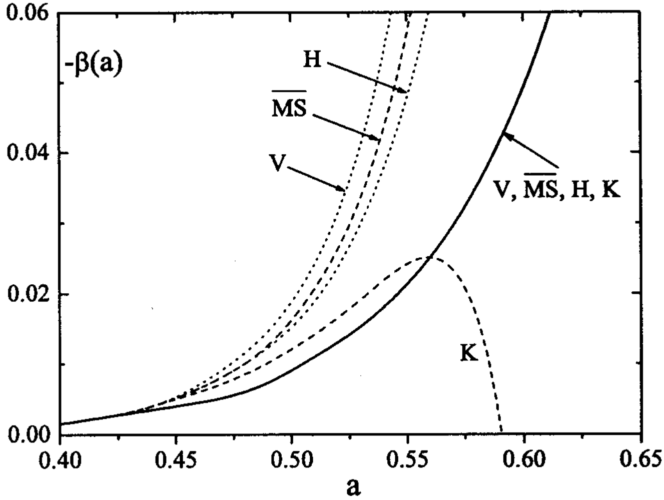

Figure 2: *RS-independence of the β-function in VPT, compared with the perturbative β-function for different renormalization schemes. The curves correspond to the VPT approach (solid) and to the PT β-function in* V *and* H *schemes (dot), in* $\overline{\mathrm{MS}}$ *and* K *schemes (dash).*

and VPT approaches is given in Fig. 2. As can be seen from graph, the differences between curves obtained in different schemes for the perturbative β-function are quite dramatic. The VPT results obtained in the same schemes practically coincide.

In Fig. 3, we demonstrate the behavior of the corresponding effective running coupling $\alpha_s(Q^2)$ in different RS's. It is seen that the uncertainties coming from the RS-dependence of perturbative calculations are rather large [see curves $\overline{\mathrm{M}}$S, V, K and for the so-called 't Hooft scheme H].

Renormalization scheme dependence for the Adler D-function. We have been found all the VPT parameters and now can study, as an important example, the renormalization scheme dependence for the Adler D-function [19]. Separating the QCD correction in the D-function we present $D = 1 + d$. The perturbation expansion for the QCD correction to the

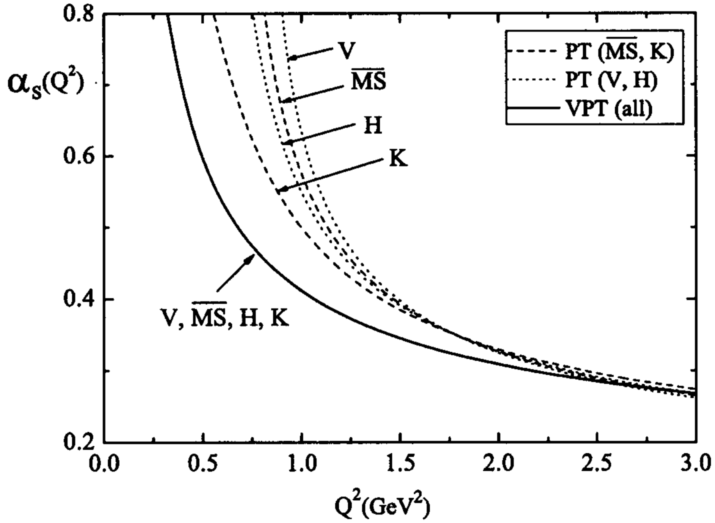

Figure 3: *The running coupling $\alpha_s(Q^2)$ calculated in the cases of perturbation theory (PT) and the variational perturbation theory (VPT) in different RS's.*

D-function with the running coupling α_s is written as:

$$d_{PT}(Q^2) = \alpha_s^i(Q^2)/\pi \left[1 + d_1^i \alpha_s^i(Q^2)/\pi + d_2^i \left(\alpha_s^i(Q^2)/\pi\right)^2 + \dots\right], \qquad (10)$$

where the index i denotes the RS in which one performs calculations. In $\overline{MS}$-scheme the perturbative coefficients for the D-function are [20, 21]

$$d_1^{\overline{MS}} = 1.986 - 0.115f\,, \qquad (11)$$

$$d_2^{\overline{MS}} = 18.244 - 4.216f + 0.086f^2 - \frac{1.2395}{3}\frac{(\sum_{f'}^{f} Q_{f'})^2}{\sum_{f'}^{f} Q_{f'}^2}\,. \qquad (12)$$

The invariant charge is determined as a solution of the renormalization group equation (2) with the three-loop β-function [17]

$$\beta(\tilde{\alpha}) = -\frac{b}{2}\,\tilde{\alpha}^2(1 + b_1\tilde{\alpha} + b_2\tilde{\alpha}^2)\,, \qquad \tilde{\alpha} \equiv \alpha_s/\pi\,, \qquad (13)$$

where

$$b = \frac{\beta_0}{2}, \qquad b_1 = \frac{4\beta_1}{\beta_0}, \qquad b_2^{\overline{MS}} = 16\,\frac{\beta_2^{\overline{MS}}}{\beta_0}, \qquad f = 3\,.$$

In passing from one renormalization scheme to another, $RS \to RS'$, the running coupling transforms as follows

$$\tilde{\alpha} \to \tilde{\alpha}' \;=\; \tilde{\alpha}(1 + v_1\tilde{\alpha} + v_2\tilde{\alpha}^2 + \ldots). \tag{14}$$

The corresponding three-loop contribution for the QCD correction to the D-function is written as

$$d \;=\; \tilde{\alpha}(1 + d_1\tilde{\alpha} + d_2\tilde{\alpha}^2)\,. \tag{15}$$

A change in the RS modifies the values of the expansion coefficients in (14) and (15). The coefficients b and b_1 are RS-independent in the class of mass and gauge independent schemes and the three-loop β-function coefficient b_2 and the expansion coefficients d_1 and d_2 in (15) depend on the choice of the renormalization scheme. Under the scheme transformation (14), they are changing and each term in representation (15) undergoes a transformation, and we obtain a new function

$$d \to d' = \tilde{\alpha}'(1 + d_1'\tilde{\alpha}' + d_2'\tilde{\alpha}'^2)\,. \tag{16}$$

Thus, in a new scheme in the VPT approach we have the corresponding three-loop contribution for the QCD correction to the D-function in the form:

$$\begin{aligned} d'_{VPT} =& \frac{4a^2}{C}\left[1 + 3\,a + \frac{2}{C}(2d_1 + 3C)\,a^2 + \frac{2}{C}(12d_1 + 5C)\,a^3\right. \\ &\left.+\ \frac{84d_1C + 15C^2 + 16d_2}{C^2}\,a^4 + \frac{224d_1C + 144d_2 + 21C^2}{C^2}\,a^5\right]. \end{aligned} \tag{17}$$

To obtain the invariant charge in the perturbative case or VPT, we use the equation

$$\frac{b}{2}\ln\left(\frac{Q^2}{\Lambda^2_{\overline{MS}}}\right) \;=\; d_1^{\overline{MS}} - d_1 + \frac{1}{\tilde{\alpha}} + b_1\ln\frac{b\tilde{\alpha}/2}{1 + b_1\tilde{\alpha}} + \Phi(\tilde{\alpha}, b_2)\,, \tag{18}$$

where

$$\Phi_{PT}(\tilde{\alpha}, b_2) = b_2\int\limits_0^{\tilde{\alpha}}\frac{dx}{(1 + b_1x)(1 + b_1x + b_2x^2)} \tag{19}$$

and

$$\Phi_{VPT}(\tilde{\alpha}, b_2) = \frac{b}{2} \int_0^{\tilde{\alpha}} \left[\frac{1}{\beta_{VPT}(x)} - \frac{1}{\beta_{PT}^{2-loop}(x)} \right] dx \,. \tag{20}$$

Although there are no general arguments to prefer a certain renormalization scheme from the start, we use a class of "natural" schemes, which look reasonable at the three-loop level that we consider. A condition for selecting a class of acceptable schemes has been proposed in [22]. One should restrict oneself to the schemes, where the cancellations between different terms in the second scheme invariant [23]

$$\rho_2 = b_2 + d_2 - b_1 d_1 - d_1^2 \tag{21}$$

are not too large. Quantitatively, this criterion can be related to the cancellation index

$$C = \frac{1}{|\rho_2|} \left(|b_2| + |d_2| + d_1^2 + |d_1| b_1 \right) \,. \tag{22}$$

Fig. 4 shows the behavior of the QCD contribution to the D-function, $d(Q^2)$, in different RS's, using for normalization the typical value of the scale parameter $\Lambda_{\overline{\rm MS}}^{(f=3)} = 370$ MeV.

The calculations were performed in the schemes A, B, which are similar to each other and to the optimal PMS and ECH schemes in the sense of the cancellation index: $C_{\rm A} \simeq C_{\rm B} \simeq C_{\rm PMS} \simeq 2$. For ECH scheme the cancellation index is minimal, equaling unity. The cancellation index for the the widely used $\overline{\rm MS}$ scheme turns out to be somewhat bigger, $C_{\overline{\rm MS}} \simeq 3.1$. In addition, we use the results for the K scheme ($C_{\rm K} \simeq 5.3$) [18] and for the V scheme ($C_{\rm V} \simeq 3.76$) [24].

As seen from Fig. 4, the dispersion of the PT results obtained for $d(Q^2)$ within different RS's diverge considerably at low energy scales as early as the value $Q^2 \simeq 2$ GeV^2. For the same schemes, in Fig. 4 we also present results obtained in the VPT approach. There the scheme arbitrariness is very small, and all the curves corresponding to the schemes A, B, ECH, $\overline{\rm MS}$, K and V calculated in VPT merge into one thick solid curve.

The one of the reason of RS-independence for the $d(Q^2)$ is the behavior of the effective coupling $\alpha_s(Q^2)$ in different renormalization schemes (Fig. 3). Thus, in the VPT, the scheme arbitrariness is very dramatically reduced as compared to that in analogous PT calculations.

Conclusions. Summarizing, we have analyzed the RS-dependence for the three-loop β-function and the effective coupling $\alpha_s(Q^2)$ in the framework

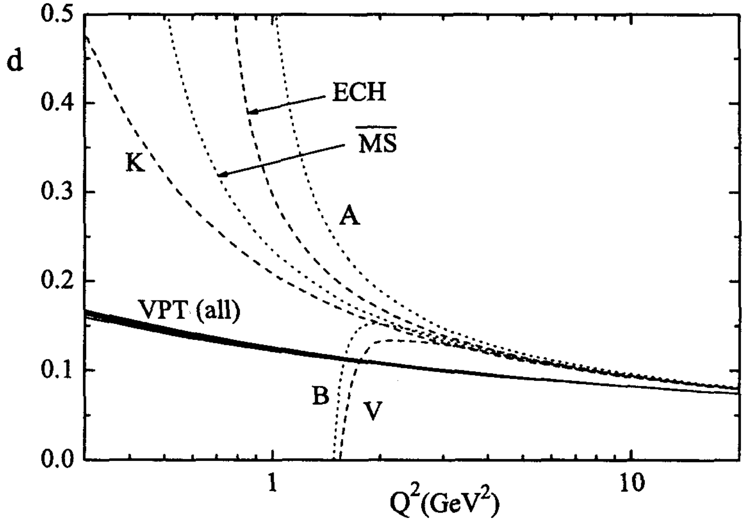

Figure 4: *Renormalization scheme dependence of the functions $d_{\mathrm{PT}}(Q^2)$ and $d_{\mathrm{VPT}}(Q^2)$ calculated in the cases of perturbation theory (dash and dot lines) and the VPT approach (solid lines) for different RS's ($f = 3$).*

of the nonperturbative approach to QCD based on the idea of variational perturbation theory. The VPT results have extraordinary stability with respect to the choice of the *RS*. Here, we have found that the QCD contribution to the *D*-function, calculated within the VPT method turned out to be practically scheme-independent in a wide class of *RS*'s. In the VPT, therefore, the three-loop level reached presently for a number of physical processes is practically invariant with respect to the choice of the renormalization prescription. Our analysis is not based on any optimization of the scheme arbitrariness(see, for example, [23]). This in turn means that the three-loop level attained for many processes is practically independent of the choice of the renormalization scheme in VPT.

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References

[1] N. N. Bogoliubov, D. V. Shirkov, Introduction to the Theory of Quantized Fields (Wiley, New York, 1959 and 1980).

[2] I. L. Solovtsov, Phys. Lett. **B 327** (1994) 335.

[3] I. L. Solovtsov, Phys. Lett. **B 340** (1994) 245.

[4] I. L. Solovtsov, Sov. Phys. J. **33**, No.7 (1990) 605-609; A. N. Sissakian and I. L. Solovtsov, Phys. Lett. **A 157** (1991) 261; Z. Phys. **C 54** (1992) 263; Phys. Part. Nucl. **25** (1994) 478; Phys. Part. Nucl. **30** (1999) 461.

[5] A. N. Sissakian, I. L. Solovtsov and O. Yu. Shevchenko, Phys. Lett. **B 297** (1992) 305; Phys. Lett. **B 313** (1993) 367; Int. J. Mod. Phys. **A 9** (1994) 1797; Int. J. Mod. Phys. **A 9** (1994) 1929.

[6] L. D. Korsun, A. N. Sissakian, I. L. Solovtsov, Theor. Math. Phys. **90** (1992) 37; Int. J. Mod. Phys. **A 8** (1994) 5129.

[7] A. N. Sissakian, I. L. Solovtsov and O. P. Solovtsova, Mod. Phys. Lett. **A 9** (1994) 2437; Phys. Lett. **B 321** (1994) 381; K. A. Milton, I. L. Solovtsov and O. P. Solovtsova, Eur. Phys. J. **C 13** (2000) 497-502.

[8] K.A. Milton, I.L.Solovtsov, O.P.Solovtsova, Phys. Rev. **D 60** 016001 (1999) 421.

[9] Milton K.A., Solovtsova O.P., Int.J.Mod.Phys. **A 17** (2002) 3789.

[10] A.N. Sisakian, I.L. Solovtsov, O.P. Solovtsova, (Dubna, JINR) 2001. Published in JETP Lett. 73:166-169,2001, Pisma Zh.Eksp.Teor.Fiz.**73** (2001) 186.

[11] D. V. Shirkov and I. L. Solovtsov, JINR Rap. Comm. **2** (1996) 5; Phys. Rev. Lett. **79** (1997) 1209; Theor. Math. Phys. **120** (1999) 1220.

[12] L. D. Korsun, I. L. Solovtsov. Proc. of the IX Intern. Seminar "Nonlinear Phenomena in Complex Systems", 16-19 May, 2000, Minsk, Belarus, eds. L. Babichev and V. Kuvshinov, B.I. Stepanov Institute of Physics National Academy of Sciences of Belarus, Minsk, (2000) 138.

[13] L. D. Korsun, I. L. Solovtsov. Proc. of Int. School-Seminar on "Actual problems of particle physics", August 2001, Gomel, Belarus, JINR E1, 2-2002-166, Dubna, (2002) v. I, 265.

[14] W. E. Caswell, Ann. Phys. **123** (1979) 153.

[15] J. Killingbeck, J. Phys. **A 14** (1981) 1005.

[16] P. M. Stevenson, Nucl. Phys. **B 231** (1984) 65.

[17] T. van Ritbergen, J. A. M. Vermaseren, S. A. Larin, Phys. Lett. **B 400** (1997) 397, hep-ph/9701390.

[18] N.V. Krasnikov and A.A. Pivovarov, Mod. Phys. Lett. **A11** (1996) 835.

[19] P. D. Adler, Phys. Rev. D.**D 10** (1974) 3714.

[20] S. G. Gorishny, A. L. Kataev and S. A. Larin, Phys. Lett. **B 259** (1991) 144.
[21] K. G. Chetyrkin, A. L. Kataev and F. V. Tkachev, Phys. Lett. **B 85** (1979) 277.
[22] P. A. Rączka, Z. Phys. **C 65** (1995) 481.
[23] P. M. Stevenson, Phys. Rev. **D 23** (1981) 2916; Phys. Lett. **B 100** (1981) 61; Nucl. Phys. **B 203** (1982) 472.
[24] W. Fischler, Nucl. Phys. **B 129** (1977) 157; T. Appelquist, M. Dine, I.J. Muzinich, Phys. Lett. **B 69** (1977) 231.