Relativistic Equations for s-Waves with Delta-Potential and Derivative of Delta-Potential

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Abstract

Exact wave functions of the scattering states of two spinless particle systems are found for delta-potential and derivative of delta-potential. Angular momentum is taken to be zero. The expressions for scattering amplitudes, S-matrix and phase shifts are calculated.

1 Introduction

In this paper we consider relativistic integral equations for wave functions of two spinless particle systems. The angular momentum of these systems is taken to be zero. These equations written in the relativistic configurational representation (RCR) are relativistic generalizations of the Schrödinger equation in the coordinate representation [1, 2]

$$\psi_q(r) = \sin(q\,r) + \int_0^\infty g_q^{(0)}(r,\ r')\,V(r')\,\psi_q(r')\,dr', \tag{1}$$

$$g_q^{(0)}(r,\ r') = \frac{-1}{q}\sin(q\,r_<)\exp(iq\,r_>). \tag{2}$$

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Here $g_q^{(0)}(r,\ r')$ is the non relativistic Green function for the scattering states. Relativistic Green functions are following expressions in the momentum representation (for four variants of the quasipotential approach in quantum field theory) [3, 4, 5]

$$G^{(1)}(\chi,\chi_q)=\frac{1}{\cosh^2\chi-\cosh^2\chi_q};G^{(2)}(\chi,\chi_q)=\frac{1}{2\cosh\chi\,(\cosh\chi-\cosh\chi_q)};\tag{3}$$

$$G^{(3)}(\chi,\chi_q)=\frac{\cosh\chi}{\cosh^2\chi-\cosh^2\chi_q};G^{(4)}(\chi,\chi_q)=\frac{1}{2(\cosh\chi-\cosh\chi_q)},$$

where χ_q is rapidity. These Green functions can be written in the RCR with the help of some integral transformation, which for s-waves has the form of the Fourier transform [5]:

$$g_q^{(j)}(r,\ r')=\frac{-2}{\pi\, m}\int\limits_0^\infty \sin(\chi\, m\, r)\, G^{(j)}(\chi,\ \chi_q)\sin(\chi\, m\, r')\, d\chi,\tag{4}$$

where $j=\overline{1,4}$ is index of the Green function's variant. In the RCR these Green functions have the following form [6, 7]:

$$g_q^{(j)}(r,\ r')=\frac{-i}{K_q^{(j)}}\{g_q^{(j)}(r-r')-g_q^{(j)}(r+r')\},\tag{5}$$

where

$$K_q^{(1)}=K_q^{(2)}=m\ \sinh(2\chi_q);\quad K_q^{(3)}=K_q^{(4)}=2m\ \sinh\chi_q,\tag{6}$$

and for $g_q^{(j)}(r)$ we have

$$g_q^{(1)}(r)=\frac{\sinh[(\pi/2+i\chi_q)m\ r]}{\sinh[\pi\ m\ r/2]},\tag{7}$$

$$g_q^{(2)}(r)=\frac{\sinh[(\pi+i\chi_q)m\ r]}{\sinh[\pi\ m\ r]}+\frac{i\ \sinh\chi_q}{2\cosh[\pi\ m\ r/2]},\tag{8}$$

$$g_q^{(3)}(r)=\frac{\cosh[(\pi/2+i\chi_q)m\ r]}{\cosh[\pi\ m\ r/2]},\tag{9}$$

$$g_q^{(4)}(r) = \frac{\sinh[(\pi + i\chi_q)m\, r]}{\sinh[\pi\, m\, r]}. \tag{10}$$

In the limit $\chi_q \to 0$ and $m \to \infty$ expressions (5-10) are transformed into non relativistic function $g_q^{(0)}(r,\ r')$ (the non relativistic momentum is $q = m\chi_q$).

For obtaining the asymptotic of solutions we need to know the behavior of Green functions for $r \to \infty$. All Green functions (5-10) have the following limiting behavior:

$$g_q^{(j)}(r,\ r')\Big|_{r\to\infty} \cong \frac{-2}{K_q^{(j)}} \sin(\chi_q m\, r') \exp(i\chi_q m\, r). \tag{11}$$

2 The solution of relativistic equations with delta-potential

The relativistic integral equations for the scattering states wave functions with zero angular momentum in RCS have the following form [6, 7]:

$$\psi_q^{(j)}(r) = \sin(\chi_q m\, r) + \int_0^\infty g_q^{(j)}(r,\ r')\, V(r')\, \psi_q^{(j)}(r')\, dr'. \tag{12}$$

Analogous one dimensional relativistic equations were considered in [8, 9].

Let us find solutions (12) for the potential

$$V(r) = V_0\, \delta(r - a), \tag{13}$$

where V_0 and a are real constants, moreover $a > 0$. The solution of (12) with (13) is given by

$$\psi_q^{(j)}(r) = \sin(\chi_q m\, r) + V_0\, g_q^{(j)}(r,\ a)\, \psi_q^{(j)}(a), \tag{14}$$

$$\psi_q^{(j)}(a) = \sin(\chi_q m\, a)\left[1 - V_0\, g_q^{(j)}(a,\ a)\right]^{-1}. \tag{15}$$

The asymptotic behavior of these wave functions may be written in the form

$$\psi_q^{(j)}(r)\Big|_{r\to\infty} \cong \sin(\chi_q m\, r) + q\, f^{(j)}(\chi_q) \exp(i\chi_q m\, r), \tag{16}$$

where we introduced the scattering amplitude $f^{(j)}(\chi_q)$, which is given by

$$f^{(j)}(\chi_q) = -\frac{2V_0 \sin(\chi_q m\, a)\, \psi_q^{(j)}(a)}{qK_q^{(j)}}. \tag{17}$$

Here $q = msh\chi_q$ is the relativistic momentum. Taking into account (15) this expression can be written in the following form:

$$f^{(j)}(\chi_q) = -\frac{2V_0 \sin^2(\chi_q m\, a)}{qK_q^{(j)}\left(1 - V_0\, g_q^{(j)}(a,\, a)\right)}, \tag{18}$$

where

$$g_q^{(1)}(a,\, a) = \frac{-i}{K_q^{(1)}} \left\{ \frac{\pi + 2i\chi_q}{\pi} - \frac{\sinh[(\pi + 2i\chi_q)m\, a]}{\sinh[\pi\, m\, a]} \right\}, \tag{19}$$

$$g_q^{(2)}(a,\, a) = \frac{-i}{K_q^{(2)}} \left\{ \frac{\pi + i\chi_q}{\pi} - \frac{\sinh[(\pi + i\chi_q)2m\, a]}{\sinh[2\pi\, m\, a]} \right\} +$$

(20)

$$+(2m\, \cosh\chi_q)^{-1} \left\{ 1 - \frac{1}{\cosh[\pi\, m\, a]} \right\},$$

$$g_q^{(3)}(a,\, a) = \frac{-i}{K_q^{(3)}} \left\{ 1 - \frac{\cosh[(\pi + 2i\chi_q)m\, a]}{\cosh[\pi\, m\, a]} \right\}, \tag{21}$$

$$g_q^{(4)}(a,\, a) = \frac{-i}{K_q^{(4)}} \left\{ \frac{\pi + i\chi_q}{\pi} - \frac{\sinh[(\pi + i\chi_q)2m\, a]}{\sinh[2\pi\, m\, a]} \right\}. \tag{22}$$

The knowledge of amplitude $f^{(j)}(\chi_q)$ gives information about scattering of particles. The scattering amplitude can be represented as follows for j equal to 3:

$$f^{(3)}(\chi_q) = \frac{-2V_0 \sin^2(\chi_q m\, a)(m\, \sinh\chi_q)^{-1}}{2m\, \sinh\chi_q + V_0\left[th(\pi m\, a)\sin(2\chi_q ma) + 2i\sin^2(\chi_q ma)\right]}. \tag{23}$$

All physical quantities, which characterize the system, can be expressed through the square of module of the scattering amplitude $\left|f^{(j)}(\chi_q)\right|^2$.

In the non relativistic theory it is important to know the S-matrix $s^{(j)}(\chi_q)$ [1, 2], which is defined by

$$f^{(j)}(\chi_q) = \frac{s^{(j)}(\chi_q) - 1}{2i\, q}. \tag{24}$$

The S-matrix is unitary, which is reflected in the following representation:

$$s^{(j)}(\chi_q) = \exp\left(2i\, \delta^{(j)}(\chi_q)\right), \tag{25}$$

where $\delta^{(j)}(\chi_q)$ is the phase shift. Now, let the relativistic S-matrix be defined also by (24). Expression for the S-matrix may be represented with the help (18), (24) and have the following form:

$$s^{(j)}(\chi_q) = \frac{K_q^{(j)}\left(1 - V_0\, g_q^{(j)}(a,\ a)\right) - 4i\, V_0 \sin^2(\chi_q m\, a)}{K_q^{(j)}\left(1 - V_0\, g_q^{(j)}(a,\ a)\right)}. \tag{26}$$

It is easy to see that in all cases $|s^{(j)}(\chi_q)| = 1$. It means that the S-matrix can be represented in the form (25). It follows from (26), that the unitarity of the S-matrix may be proved on the base of the following property of Green functions:

$$\mathrm{Im}(g_q^{(j)}(a,\ a)) = -\frac{2\sin^2(\chi_q m\, a)}{K_q^{(j)}}. \tag{27}$$

The expression for the S-matrix for $j = 3$ has the form:

$$s^{(3)}(\chi_q) = \frac{\sinh\chi_q + (\frac{V_0}{2m})\left[th(\pi\ m\ a)\sin(2\chi_q m\ a) - 2\, i \sin^2(\chi_q m a)\right]}{\sinh\chi_q + (\frac{V_0}{2m})\left[th(\pi\ m\ a)\sin(2\chi_q m\ a) + 2\, i \sin^2(\chi_q m a)\right]}. \tag{28}$$

The other expressions for the scattering amplitudes and the S-matrix are more complicated.

It is easy to obtain the expressions for the phase shift. So for example from (25) and(28) we have:

$$tg2\delta^{(3)} = \frac{-2(\frac{V_0}{m})\sin^2(\chi_q m a)\left[\ \sinh\chi_q + (\frac{V_0}{2m}) th(\pi m a)\sin(2\chi_q m a)\right]}{\left[\ \sinh\chi_q + \frac{V_0}{2m} th(\pi m a)\sin(2\chi_q m a)\right]^2 - (\frac{V_0}{m})^2 \sin^4(\chi_q m a)}. \tag{29}$$

The other phase shifts are also found and used at numerical calculations.

In figures 1-4 the square of module of the scattering amplitude and phase shift are given as functions of rapidity for parameters m, a, V_0 fixed.

3 The solution of relativistic equations with derivative of delta-potential

Now let us consider equations (12) for the potential

$$V(r) = V_1\, \delta\,'(r-a), \tag{30}$$

where V_1 and a are real constants, and again $a > 0$. The solution of equations (12) with this potential is

$$\psi_q^{(j)}(r) = \sin(\chi_q m\, r) - V_1\, [\, g_q^{(j)}\,'(r,\, a)\, \psi_q^{(j)}(a) + g_q^{(j)}(r,\, a)\, \psi_q^{(j)}\,'(a)], \tag{31}$$

$$\psi_q^{(j)}(a) = \frac{\Delta_{q1}^{(j)}}{\Delta_q^{(j)}}; \quad \psi_q^{(j)}\,'(a) = \frac{\Delta_{q2}^{(j)}}{\Delta_q^{(j)}}, \tag{32}$$

where

$$\Delta_q^{(j)} = [1 + V_1\, g_q^{(j)}\,'(a,\, a)]^2 - V_1^2 g_q^{(j)}(a,\, a)\, g_q^{(j)}\,''(a,\, a);$$

$$\Delta_{q1}^{(j)} = \sin(\chi_q m\, a)[1 + V_1\, g_q^{(j)}\,'(a,\, a)] - V_1\, \chi_q\, g_q^{(j)}(a,\, a) \cos(\chi_q m\, a); \tag{33}$$

$$\Delta_{q2}^{(j)} = \chi_q \cos(\chi_q m\, a)[1 + V_1\, g_q^{(j)}\,'(a,\, a)] - V_1\, g_q^{(j)}\,''(a,\, a) \sin(\chi_q m\, a).$$

Here we use the following notations:

$$g_q^{(j)}\,'(r,\, a) = \frac{\partial}{\partial r'} g_q^{(j)}(r,\, r')\bigg|_{r\,'=a} ;\, g_q^{(j)}\,''(r,\, a) = \frac{\partial^2}{\partial r \partial r'} g_q^{(j)}(r,\, r')\bigg|_{r\,'=a} . \tag{34}$$

We can write the asymptotic formulae of wave function for $r \to \infty$ in the following form:

$$\psi_q^{(j)}(r)\big|_{r\to\infty} \cong \sin(\chi_q m\, r) + q\, f^{(j)}(\chi_q) \exp(i \chi_q m\, r), \tag{35}$$

where the scattering amplitudes for the derivative of delta-potential are

$$f^{(j)}(\chi_q) = \frac{2V_1}{q K_q^{(j)} \Delta_q^{(j)}} \left[\, \chi_q \cos(\chi_q m\, a)\, \Delta_{q1}^{(j)} + \sin(\chi_q m\, a)\, \Delta_{q2}^{(j)} \right]. \tag{36}$$

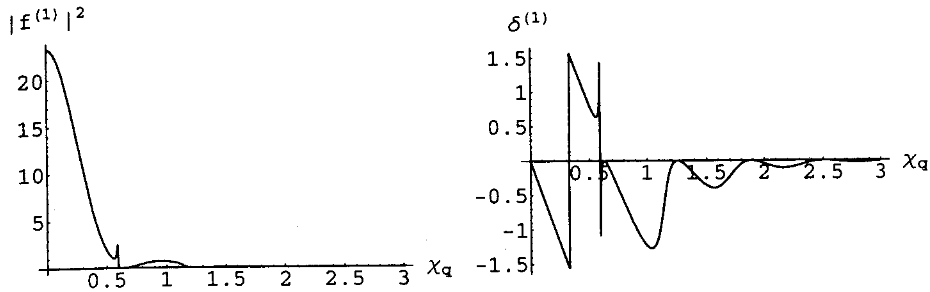

Figure 1: Square of module of the scattering amplitude and phase shift as functions of rapidity for $m = 1$, $V_0 = 2$, $a = 5$ for $j = 1$.

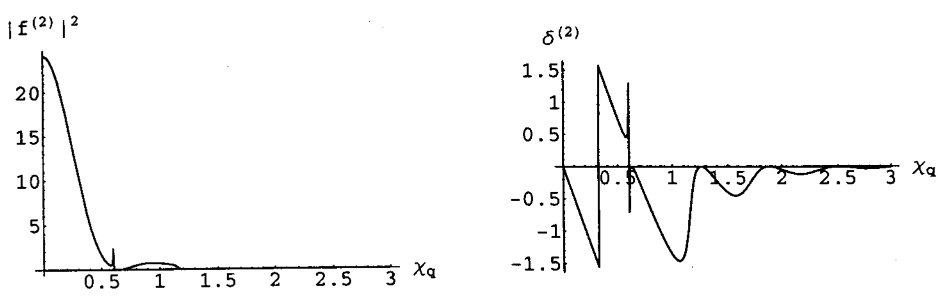

Figure 2: Square of module of the scattering amplitude and phase shift as functions of rapidity for $m = 1$, $V_0 = 2$, $a = 5$ for $j = 2$.

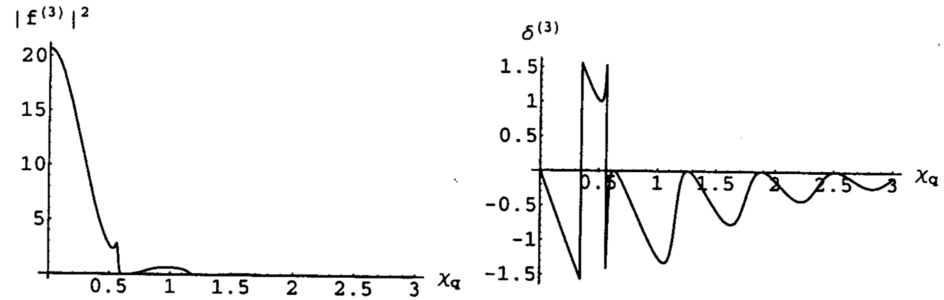

Figure 3: Square of module of the scattering amplitude and phase shift as functions of rapidity for $m = 1$, $V_0 = 2$, $a = 5$ for $j = 3$.

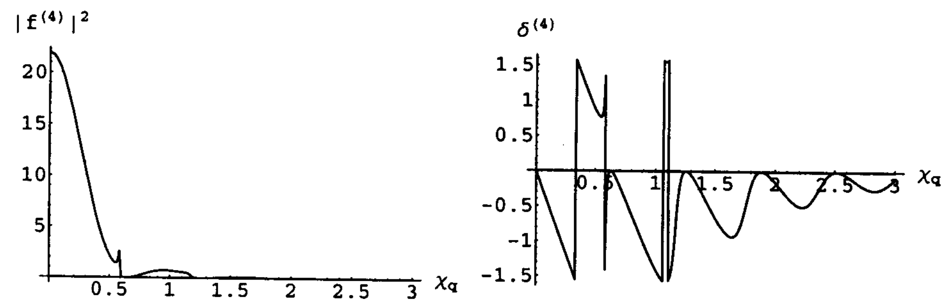

Figure 4: Square of module of the scattering amplitude and phase shift as functions of rapidity for $m = 1$, $V_0 = 2$, $a = 5$ for $j = 4$.

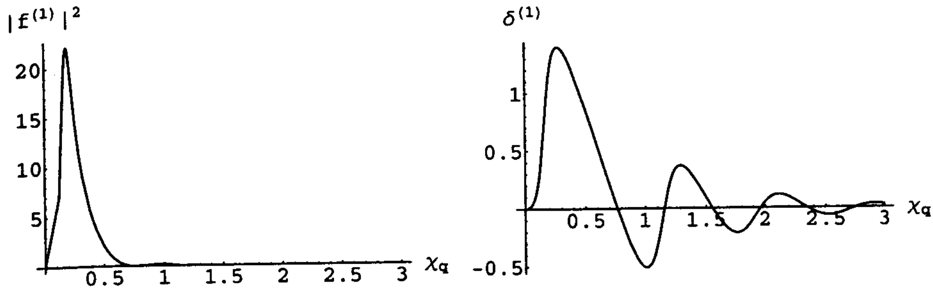

Figure 5: Square of module of the scattering amplitude and phase shift as functions of rapidity for $m = 1$, $V_0 = -1$, $a = 4$ for $j = 1$.

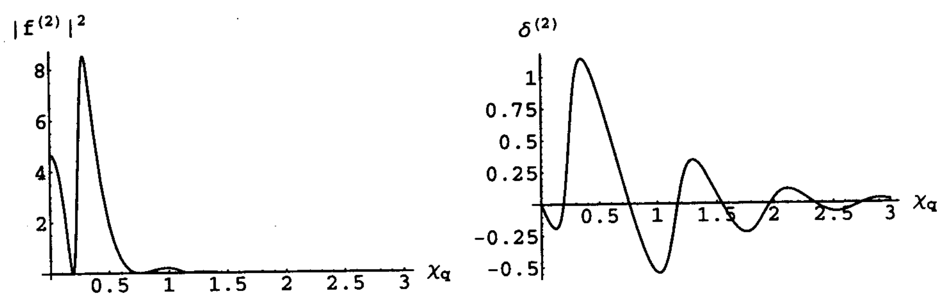

Figure 6: Square of module of the scattering amplitude and phase shift as functions of rapidity for $m = 1$, $V_0 = -1$, $a = 4$ for $j = 2$.

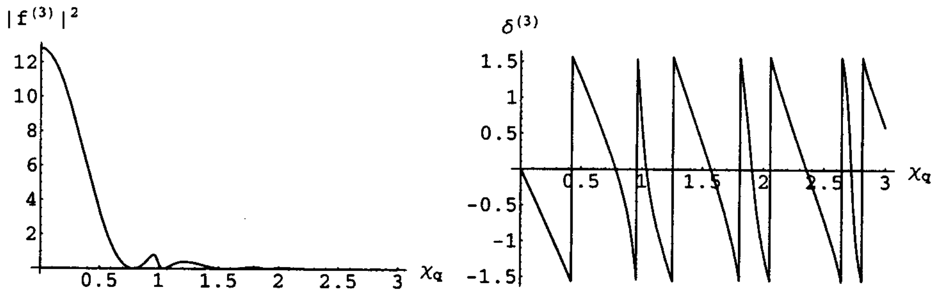

Figure 7: Square of module of the scattering amplitude and phase shift as functions of rapidity for $m = 1$, $V_0 = -1$, $a = 4$ for $j = 3$.

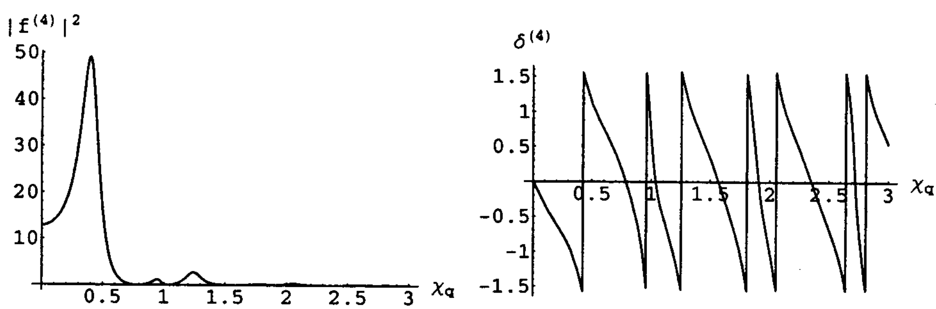

Figure 8: Square of module of the scattering amplitude and phase shift as functions of rapidity for $m = 1$, $V_0 = -1$, $a = 4$ for $j = 4$.

These expressions for the scattering amplitude, which characterize the scattering for the concrete Green function here are cumbersome and because of this reason are not written here, but they are used in numerical calculations. In figures 5-8 the square of module of the scattering amplitude and phase shift are given as functions of rapidity for parameters m, a, V_1 fixed.

4 Conclusion

Thus, in this article we solved exactly relativistic two particle equations with singular potentials $V_0\ \delta(r-a)$ and $V_1\ \delta'(r-a)$ in the spherically symmetrical case. Exact solutions allowed us to find the partial S-matrix, the phase shift and resonant structure of these quantities. The possibility to solve relativistic equations with the potential $V_1\ \delta'(r-a)$ means, that in the relativistic case this potential is less singular than in the non relativistic one. So, the transition from the Schrödinger equation to the relativistic equations is a natural physical regularization for the potential $V_1\ \delta'(r-a)$.

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