Relativistic Inverse Scattering Problem for a Sum of a Nonlocal Separable Quasipotentials

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Abstract

A relativistic inverse problem is solved for the case when the total quasipotential simulating the interaction of two relativistic spinless particles of unequal masses is the superposition of a local quasipotential and a sum of nonlocal separable quasipotentials. The problem is investigated within the relativistic quasipotential approach to quantum field theory. The local component of total interaction is supposed to be known and it not admits bound states. It is shown that the nonlocal separable components of total interaction may be reconstructed if its the local component, the phase-shift additions and the true bound state energy are known.

The inverse scattering problem has a long history. So, it was proven by Gelfand and Levitan [1], Marchenko [2], and Krein [3] that the inverse problem can be solved within nonrelativistic theory. The most complete survey of this theory was given in the monographs of Chadan and Sabatier [4] and Zakhariev and Suzko [5]. In the most of studies, however, the inverse problem is solved on the basis of the nonrelativistic Schrödinger equation [6–10]. Therefore, the problem of reconstructing interaction for essentially relativistic systems – in particular, within the relativistic quasipotential approach [11] – is yet remained important.

In the present study, the problem of reconstructing the nonlocal separable components of the total quasipotential describing the interaction between

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two relativistic spinless particles of unequal masses ($m_1 \neq m_2$) is considered. The problem is investigated within the relativistic quasipotential approach to quantum field theory [12]. It is assumed that the total interaction admits the existence of the only true bound state. Moreover, the local part $W(r)$ of the total interaction is considered to be known and it not admits bound states. We will show that the nonlocal separable components $V_{ln}(r)$ of the total interaction can be reconstructed provided that the local part $W(r)$, the additions of the phase shift $\delta_l^{V_{ln}}(\chi')$, and the energies of the bound states are known. Our approach is based on the expression for the additions of the phase shift ($\hbar = c = 1$)

$$\tan\delta_l^{V_{ln}}(\chi') = -\frac{\pi}{2}\sinh^{-1}(\chi')A_{ln}(\chi')\left[1+\frac{1}{2}\mathrm{P}\int\limits_0^\infty d\chi\frac{A_{ln}(\chi)}{\cosh\chi-\cosh\chi'}\right]^{-1}, \quad (1)$$

$$A_{ln}(\chi) = \frac{2}{\pi}\varepsilon_{ln}\left|Q_l(\coth\chi)\tilde{V}_{ln}^{(n-1)}(\chi)/F_l^W(\chi)\right|^2\prod_{m=1}^{n-1}\left[\cos\delta_l^{V_{lm}}(\chi)\right]^2, \quad (2)$$

$$\varepsilon_{ln} = \pm 1, \ n = 1, 2, ..., M_l,$$

where P means the principal value, $Q_l(z)$ is a Legender function of the second kind, $\tilde{V}_{ln}^{(n-1)}(\chi)$ is the transform of the components $V_{ln}(r)$, and $F_l^W(\chi)$ is the Jost function of the local quasipotential $W(r)$ and is related to the corresponding phase shift $\delta_l^W(\chi)$ by the equation

$$F_l^W(\chi) = \left|F_l^W(\chi)\right| \ \exp\left[-i\delta_l^W(\chi)\right].$$

In order to reconstruct the separable component $V_{ln}(r)$ on the basis of the additions of the phase shift $\delta_l^{V_{ln}}(\chi')$, we will solve the integral Eq.(1) for the function $A_{ln}(\chi')$. Here we will use the results that were obtained by the author in [13]. After that, by using of the Hilbert integral transformation, we find the function $\tilde{V}_{ln}^{(n-1)}(\chi')$ from (2). Finally, by performing the relativistic Hankel transformations

$$V_{ln}(r) = \int\limits_1^\infty d\rho_l^{(n-1)}(\cosh\chi)\tilde{V}_{ln}^{(n-1)}(\chi)\psi_l^{(n-1)}(r,\chi), \ n = 1, 2, ..., M_l, \quad (3)$$

we reconstruct the components $V_{ln}(r)$. Here, $\psi_l^{(n-1)}(r,\chi)$ is the solution to the finite-difference quasipotential equation for the superposition of a local quasipotential $W(r)$ and a sum of nonlocal separable quasipotentials $\sum\limits_{m=1}^{n-1} \varepsilon_{lm} V_{lm}(r) V_{lm}(r')$. This the solution will correspond the spectral density

$$\frac{d\rho_l^{(n-1)}(\cosh\chi)}{d(\cosh\chi)} = \frac{d\rho_l^{(0)}(\cosh\chi)}{d(\cosh\chi)} \prod_{m=1}^{n-1} \left[\cos\,\delta_l^{V_{lm}}(\chi)\right]^2 \;,\; n=1,2,...,M_l, \qquad (4)$$

where

$$\frac{d\rho_l^{(0)}(\cosh\chi)}{d(\cosh\chi)} = \frac{2}{\pi}\sinh^{-1}(\chi)\left|Q_l(\coth\chi)/F_l^W(\chi)\right|^2$$

is the spectral density associated with the local quasipotential $W(r)$. Besides, the solution $\psi_l^{(n-1)}(r,\chi)$ will satisfy the completeness property

$$\int\limits_1^\infty d\rho_l^{(n-1)}(\cosh\chi)\psi_l^{(n-1)}(r,\chi)\psi_l^{(n-1)^*}(r',\chi) = \delta(r'-r) \;,\; n=1,2,...,M_l. \quad (5)$$

Note that, at $n=1$ the integral transformation (3) and the property (5) reduce to the corresponding expressions obtained in [14].

For the unique solution of the inverse scattering problem to exist, we assume that the additions of the phase shift $\delta_l^{V_{ln}}(\chi')$ in expression (1) is a function continuous in the sence of Hölder with a positive index and that, for $\chi' \to +\infty$, we have estimates

$$\delta_l^{V_{ln}}(\chi') = O(\chi'^{-\gamma}) \;,\; l \geq 0 \;,\; \gamma > 1 \;,\; n=1,2,...,M_l. \qquad (6)$$

These requirements means that the components $V_{ln}(r)$ of the separable interaction satisfies the conditions

$$rV_{ln}(r) \in L_1(0,\infty) \;,\; n=1,2,...,M_l. \qquad (7)$$

Besides, the Levinson theorem for the additions of the phase shift takes the form

$$\delta_l^{V_{ln}}(0) - \delta_l^{V_{ln}}(\infty) = \delta_l^{V_{ln}}(0) = \pi(\sigma_l^{(n)} - \sigma_l^{(n-1)} + \nu_l^{(n)}) \;,\; n=1,2,...,M_l \;, \quad (8)$$

where

$$\delta_l^W(0) - \delta_l^W(\infty) = \delta_l^W(0) = 0.$$

Here, $\sigma_l^{(n)}$ is the number of true bound states corresponding to the total interaction and having energies that satisfy the conditions

$$0 \leq E_t^{(n)} = \cosh\chi_t^{(n)} < 1 \ , \ \chi_t^{(n)} = i\kappa_t^{(n)}, \ 0 < \kappa_t^{(n)} \leq \pi/2 \ , \ n = 1, 2, ..., M_l, \tag{9}$$

it is known that

$$\sigma_l^{(n)} = \begin{cases} 0 \ , \ \varepsilon_{ln} = 1 \ , \ n = 1, 2, ..., M_l \ ; \\ 0 \ , \ \varepsilon_{ln} = -1 \ , \ n = 1, 2, ..., M_l - 1 \ ; \\ 1 \ , \ \varepsilon_{lM_l} = -1 \ , \ n = M_l \ . \end{cases} \tag{10}$$

At the same time $\nu_l^{(n)}$ is the number of spurious bound states associated with the component of $V_{ln}(r)$ whose energies satisfy the condition

$$E_{fk}^{(n)} = \cosh\chi_{fk}^{(n)} \geq 1 \ , \ n = 1, 2, ..., M_l \ , \ k = \begin{cases} 0, 1, ..., \nu_l^{(n)} - 1 \ , \ \varepsilon_{ln} = 1, \\ 1, 2, ..., \nu_l^{(n)}, \ \varepsilon_{ln} = -1. \end{cases} \tag{11}$$

Moreover, the energies (11) of spurious bound states can be determined from the values of χ' at which the additions of the phase shift intersects the straight lines $\delta_l^{V_{ln}} = \pi k$ (k is an integer) from above as χ' increases; that is,

$$\delta_l^{V_{ln}}(\chi_{fk}^{(n)}) = \pi k \ , \ n = 1, 2, ..., M_l \ , \ k = \begin{cases} 0, 1, ..., \nu_l^{(n)} - 1 \ , \ \varepsilon_{ln} = 1 \ , \\ 1, 2, ..., \nu_l^{(n)} \ , \ \varepsilon_{ln} = -1 \ , \end{cases} \tag{12}$$

whereas the energy (9) of the only true bound state is simple root of the equation

$$\Phi_{lM_l}(E_t^{(M_l)}) = -1 + \frac{1}{2} \int_0^\infty d\chi \frac{|A_{lM_l}(\chi)|}{\cosh\chi - E_t^{(M_l)}} = 0.$$

The integral equation (1) we recast into the form

$$\psi_{ln}(x) = 1 + \frac{1}{\pi}\int_1^\infty dt \frac{\psi_{ln}(t)h^*_{ln}(t)}{t - x - i0} \quad , \; n = 1, 2, ..., M_l, \tag{13}$$

where $x = \cosh\chi'$, $t = \cosh\chi$ and where we introduced the following notation

$$\Delta_l^{V_{ln}}(x) = \delta_l^{V_{ln}}(\text{arcosh } x) \; , \; g_{ln}(x) = -(2/\pi)(x^2-1)^{1/2}\tan\Delta_l^{V_{ln}}(x), \tag{14}$$

$$h_{ln}(x) = -\sin \; \Delta_l^{V_{ln}}(x) \exp \; \left[-\, i\Delta \;_l^{V_{ln}}(x) \right] \; ,$$

$$\psi_{ln}(x) = A_{ln}(\text{arcosh } x) g_{ln}^{-1}(x) \left[1 + i(\pi/2) g_{ln}(x)(x^2-1)^{-1/2} \right] \; .$$

As in the case of one component separable quasipotential ($n = 1$) [13], we search the solution of equation (13) in the form

$$\psi_{ln}(x) = H_{ln}(x_+) \equiv \lim_{\eta\to+0} H_{ln}(x + i\eta) \; , \; 1 \leq x \leq \infty \; , \tag{15}$$

where the function

$$H_{ln}(z) = 1 + \frac{1}{\pi}\int_1^\infty dt \frac{\psi_{ln}(t)h^*_{ln}(t)}{t - z} \tag{16}$$

is an analytic function in the complex plane of the variable z with a cut from 1 to $+\infty$. Also, we have

$$\lim_{|z|\to\infty} H_{ln}(z) = 1 \tag{17}$$

along any direction provided that the function $\psi_{ln}(x)$ is Hölder continuous and that the integral in (16) converges. Moreover, by substituting the solution in (15) into the expression for the discontinuity of the function $H_{ln}(z)$ across cut, that is

$$H_{ln}(x_+) - H_{ln}(x_-) = -2i \sin \Delta_l^{V_{ln}}(x) \exp \left[i\Delta_l^{V_{ln}}(x) \right] \; \psi_{ln}(x),$$

we arrive at the homogeneous Riemann–Hilbert equation for the function $H_{ln}(z)$:

$$H_{ln}(x_+)\exp\left[2i\Delta_l^{V_{ln}}(x)\right]-H_{ln}(x_-)=0\ ,\ 1\le x\le\infty\ ,\ n=1,2,...,M_l. \tag{18}$$

A particular solution satisfying Eq.(18) and the condition (17) is given by

$$\tilde{H}_{ln}(z)=\exp\left[\omega_{ln}(z)\right]\ , \tag{19}$$

where

$$\omega_{ln}(z)=-\frac{1}{\pi}\int_1^\infty dt\frac{\Delta_l^{V_{ln}}(t)}{t-z}\ . \tag{20}$$

From the above assumptions on the behaviour of the additions of the phase shift and from conditions (6) and (17), it follows that

$$\lim_{|z|\to\infty}\omega_{ln}(z)=0$$

along all directions. Moreover, the function in (20) is defined everywhere on the cut, with the exception of may be the point $z=1$, where its behaviour is given by

$$\omega_{ln}(z)=(1/\pi)\Delta_l^{V_{ln}}(1)\ln|1-z|+\Omega_{ln}(z)\ ,\ z\to 1. \tag{21}$$

Here, the function $\Omega_{ln}(z)$ is finite for $z\to 1$, while $\Delta_l^{V_{ln}}(1)=\delta_l^{V_{ln}}(0)=\pi(\sigma_l^{(n)}-\sigma_l^{(n-1)}+\nu_l^{(n)})$ according to the Levinson theorem (8). Therefore, the function $\tilde{H}_{ln}(z)$ has a zero of order $\sigma_l^{(n)}-\sigma_l^{(n-1)}+\nu_l^{(n)}$ at the point $z=1$. For this reason, according to the expressions (15) , (19) and (20), a particular solution has the form

$$\tilde{\psi}_{ln}(x)=\exp\left[\alpha_{ln}(x)-i\Delta_l^{V_{ln}}(x)\right], \tag{22}$$

where

$$\alpha_{ln}(x)=-\frac{1}{\pi}\mathrm{P}\int_1^\infty dt\frac{\Delta_l^{V_{ln}}(t)}{t-x}\ ,\quad \omega_{ln}(x_\pm)=\alpha_{ln}(x)\mp i\Delta_l^{V_{ln}}(x). \tag{23}$$

Note that, the function given by (22) is regular at $x = 1$ (it has a zero of order $\sigma_l^{(n)} - \sigma_l^{(n-1)} + \nu_l^{(n)}$ at this point), is continuous in the sense Hölder with the same index as the additions of phase shift, and is bounded for $x \to +\infty$ in accordance with a priori assumptions on its properties. Moreover, the function in (22) satisfies Eq.(13), since the residue theorem gives

$$\lim_{\substack{R\to+\infty \\ \eta\to+0}} \frac{1}{2\pi i} \int\limits_{\Gamma^+} dz \frac{\tilde{H}_{ln}(z)}{z - x - i\eta} = \mathrm{res}\left[\frac{\tilde{H}_{ln}(z)}{z - x - i\eta}\ , \ z = x + i\eta\right] |_{\eta\to+0}\ ,$$

where Γ^+ is a closed contour consisting of circle C_R^+ having a radius R and a center at the point $z = 0$, a circle C_η^- having a radius η and a center at the point $z = 1$, and the two banks of the cut from 1 to R, the direction of the contour along the upper bank being opposite to that along the lower bank. According to the asymptotic formula (17), the contribution of the integral along the circle C_R^+ tends to unity as $R \to +\infty$, while, according to the estimate in (21), the integral along the circle C_η^- tends to zero as $\eta \to +0$. Hence, it follows that function in (22) is a particular solution of the nonhomogeneous integral equation (13).

A general solution to the homogeneous equation

$$\psi_{lno}(x) = \frac{1}{\pi} \int\limits_1^\infty dt \frac{\psi_{lno}(t) h_{ln}^*(t)}{t - x - i0}\ , \tag{24}$$

has the form

$$\psi_{lno}(x) = H_{lno}(x_+) \equiv \lim_{\eta\to+0} H_{lno}(x + i\eta)\ ,\ 1 \le x \le \infty,$$

where the function

$$H_{lno}(z) = \frac{1}{\pi} \int\limits_1^\infty dt \frac{\psi_{lno}(t) h_{ln}^*(t)}{t - z}$$

is analytic in the complex plane of z with a cut from 1 t $+\infty$ and which satisfies the condition

$$\lim_{|z|\to\infty} H_{lno}(z) = 0$$

in all directions and also it satisfies the Riemann–Hilbert homogeneous equation (18). Therefore, a general solution to Eq.(24) will be sought in the form

$$H_{lno}(z) = \exp\left[\omega_{ln}(z)\right] \sum_{k=1}^{N_l^{(n)}} \frac{A_k^{(n)}}{(z-1)^k} \, . \tag{25}$$

Substituting (25) into (18) and requiring that the function $H_{lno}(z)$ be finite at $z = 1$ (it has a zero of order $\sigma_l^{(n)} - \sigma_l^{(n-1)} + \nu_l^{(n)}$at this point), gives $N_l^{(n)} = \sigma_l^{(n)} - \sigma_l^{(n-1)} + \nu_l^{(n)}$. Hence, we have

$$\psi_{lno}(x) = \exp\left[\alpha_{ln}(x) - i\Delta_l^{V_{ln}}(x)\right] \sum_{k=1}^{N_l^{(n)}} \frac{A_k^{(n)}}{(x-1)^k} \, , \; N_l^{(n)} = \sigma_l^{(n)} - \sigma_l^{(n-1)} + \nu_l^{(n)} . \tag{26}$$

It is obvious that, in just the same way as in the case of particular solution, integration along the contour $^+$ leads to the conclusion that the function in (26) is a solution to Eq.(24) and possesses all the required properties. Therefore, a general solution to the integral equation (13) has a form

$$\psi_{ln}(x) = \tilde{\psi}_{ln}(x) + \psi_{lno}(x) = \exp\left[\alpha_{ln}(x) - i\Delta_l^{V_{ln}}(x)\right] \left\{1 + \sum_{k=1}^{N_l^{(n)}} \frac{A_k^{(n)}}{(x-1)^k}\right\} . \tag{27}$$

Finally, by using the notation in (14) and rearranging the sum into a product, we can recast the solution in (27) into the form

$$A_{ln}(\chi') = -\frac{2}{\pi} \sinh\chi' \sin\delta_l^{V_{ln}}(\chi') \exp\left[\alpha_{ln}(\cosh\chi')\right] \prod_{k=1-\delta}^{N_l^{(n)}-\delta} \left[1 + \frac{a_k^{(n)}}{\cosh\chi' - 1}\right] , \tag{28}$$

$$\alpha_{ln}(\cosh\chi') = -\frac{1}{\pi} \mathrm{P} \int_0^\infty d\chi \frac{\sinh\chi \delta_l^{V_{ln}}(\chi)}{\cosh\chi - \cosh\chi'} , \tag{29}$$

$$N_l^{(n)} = \sigma_l^{(n)} - \sigma_l^{(n-1)} + \nu_l^{(n)} = \\ = \begin{cases} \nu_l^{(n)}, \ \varepsilon_{ln} = 1 \ , \ n = 1, 2, ..., M_l, \\ \nu_l^{(n)}, \ \varepsilon_{ln} = -1 \ , \ n = 1, 2, ..., M_l - 1 \ , \ \delta = \begin{cases} 1 \ , \ \varepsilon_{ln} = 1, \\ 0 \ , \ \varepsilon_{ln} = -1. \end{cases} \\ \nu_l^{(M_l)} + 1 \ , \ \varepsilon_{lM_l} = -1 \ , \ n = M_l \ , \end{cases} \tag{30}$$

In order to determine the parameters $\left\{ a_k^{(n)} \right\}$, we note that, by definition (2), the function $A_{ln}(\chi')$ is of fixed sign at all values of χ', whereas the additions of the phase shift at the energies (11) of spurious bound state satisfies the conditions in (12). Hence, the function $A_{ln}(\chi')$ retains of fixed sign, provided that

$$a_k^{(n)} = 1 - \cosh\chi_{fk}^{(n)} \ , \ k = \begin{cases} 0, 1, ..., \nu_l^{(n)} - 1 \ , \ \varepsilon_{ln} = 1 \ , \ n = 1, 2, ..., M_l, \\ 1, 2, ..., \nu_l^{(n)}, \ \varepsilon_{ln} = -1 \ , \ n = 1, 2, ..., M_l, \end{cases}$$

$$a^{(M_l)}_{\nu_l^{(M_l)}+1} = 1 - \cosh\chi_t^{(M_l)}, \ \sigma_l^{(M_l)} = 1 \ , \ \varepsilon_{lM_l} = -1 \ , \ n = M_l.$$

Thus, the coefficients $\left\{ a_k^{(n)} \right\}$are completely determined by the energy (9) of true bound state of the total interaction ($n = M_l$) and the additions of the phase shift, since the value $\chi_{fk}^{(n)}$ are also determined by its behaviour – that is, the conditions in (12). Instead of (28), we will then have

$$A_{ln}(\chi') = -\frac{2}{\pi}\sinh\chi' \sin\delta_l^{V_{ln}}(\chi') \exp\left[\alpha_{ln}\left(\cosh\chi'\right)\right] \ \times$$

$$\times \prod_{k=1-\delta}^{\nu_l^{(n)}-\delta} \left[\frac{\sinh^2(\chi'/2) - \sinh^2(\chi_{fk}^{(n)}/2)}{\sinh^2(\chi'/2)} \right] \left[\frac{\sinh^2(\chi'/2) + \sin^2(\kappa_t^{(n)}/2)}{\sinh^2(\chi'/2)} \right]^{\sigma_l^{(n)}}, \tag{31}$$

$$n = 1, 2, ..., M_l,$$

where $\kappa_t^{(n)}$ and $\sigma_l^{(n)}$are defined in (9) and (10), and δ is defined in (30). Moreover, it follows from expressions (29) and (31) that the functions $A_{ln}(\chi')$

are continuous in the sence of Hölder and that, for $\chi' \to +\infty$, them behaves as

$$\cosh\chi' \, |\chi'|^{-\gamma} \, , \; \gamma > 1 \; , \; n = 1, 2, ..., M_l,$$

provided that the additions of the phase shift satisfies the conditions in (6). This in his turn implies that the components $V_{ln}(r)$ satisfies conditions in (7).

In order to reconstruct the components $V_{ln}(r)$ by means of the transformation in (3), we introduce the functions

$$\hat{V}_{ln}(\sinh(\chi'/2)) = \left[\frac{\sinh(\chi'/2) + i\sin(\kappa_t^{(n)}/2)}{\sinh(\chi'/2) - i\sin(\kappa_t^{(n)}/2)}\right]^{\sigma_l^{(n)}} \times \tag{32}$$

$$\times \left[Q_l(\coth\chi') \prod_{m=1}^{n-1} \cos\left[\delta_l^{V_{lm}}(\chi')\right] \; \tilde{V}_{ln}^{(-)}(\sinh(\chi'/2)) / \left|F_l^W(\chi')\right|\right]^2 ,$$

where

$$\left|\tilde{V}_{ln}^{(-)}(\sinh(\chi'/2))\right| = \left|\tilde{V}_{ln}^{(n-1)}(\chi')\right| \; , \; \mathrm{Re}\tilde{V}_{ln}^{(-)}(\sinh(\chi'/2)) = \mathrm{Re}\tilde{V}_{ln}^{(n-1)}(\chi') \; ,$$
$$\arg \tilde{V}_{ln}^{(-)}(-\sinh(\chi' \,/\, 2)) = -\arg \tilde{V}_{ln}^{(-)}(\sinh(\chi'/2)) \; , \; n = 1, 2, ..., M_l. \tag{33}$$

Taking into account the conditions in (33), the relation $\arg\tilde{V}_{ln}^{(n-1)}(-\chi') = \arg\tilde{V}_{ln}^{(n-1)}(\chi')$ implies that

$$\arg\tilde{V}_{ln}^{(n-1)}(\chi') = \mathrm{sgn}\chi' \cdot \arg\tilde{V}_{ln}^{(-)}(\sinh(\chi'/2)). \tag{34}$$

The function $\hat{V}_{ln}(\sinh(\chi'/2))$ are then analytic in the band $0 < \mathrm{Im}\chi' \leq \pi/2$ are continuous for $0 \leq \mathrm{Im}\chi' \leq \pi/2$, and for them has the place estimate

$$\hat{V}_{ln}(\sinh(\chi'/2)) = O\left[\sinh^2(\chi'/2)\right] \; , \; |\chi'| \to \infty \; , \; 0 \leq \mathrm{Im}\chi' \leq \pi/2 \; , \tag{35}$$

if only the conditions in (6) hold. Moreover, the functions $\hat{V}_{ln}(\sinh(\chi'/2))$ does not vanish anywhere in the band $0 < \mathrm{Im}\chi' \leq \pi/2$. Therefore, the functions $\ln\hat{V}_{ln}(\sinh(\chi'/2))$ are analytic in the band $0 < \mathrm{Im}\chi' \leq \pi/2$ and, according to the estimate in (35), behaves as $\ln(\sinh^2(\chi'/2))$ when $|\chi'| \to \infty$.

Therefore, the Hilbert integral transformation can be applied to both the real and the imaginary part of the function $\ln \hat{V}_{ln}(\sinh(\chi'/2))$. For real values of χ', we then obtain

$$\mathrm{Im}\ln \hat{V}_{ln}(\sinh(\chi'/2)) = -\frac{1}{\pi}\mathrm{P}\int\limits_{-\infty}^{\infty} d(\sinh(\chi/2))\frac{\mathrm{Re}\ln \hat{V}_{ln}(\sinh(\chi/2))}{\sinh(\chi/2)-\sinh(\chi'/2)} =$$
$$= -\frac{2\sinh(\chi'/2)}{\pi}\mathrm{P}\int\limits_{0}^{\infty} d\chi\frac{\cosh(\chi/2)\ln\left[\pi\varepsilon_{ln}A_{ln}(\chi)/2\right]}{\cosh\chi-\cosh\chi'} ,$$

where we have considered that

$$\mathrm{Re}\ln \hat{V}_{ln}(\sinh(\chi/2)) = \ln\left[\pi\varepsilon_{ln}A_{ln}(\chi)/2\right] .$$

From here, taking into account the expressions (32) – (34), we finally obtain

$$|Q_l(\coth\chi')/F_l^W(\chi')|\prod_{m=1}^{n-1}\cos\left[\delta_l^{V_{lm}}(\chi')\right]\ \tilde{V}_{ln}^{(n-1)}(\chi') = \sqrt{\pi\varepsilon_{ln}A_{ln}(\chi')/2}\ \times \tag{36}$$
$$\times\exp\left\{-i\,\mathrm{sgn}\chi'\left[\sigma_l^{(n)}\arctan\frac{\sin(\kappa_t^{(n)}/2)}{\sinh(\chi'/2)}+\right.\right.$$
$$\left.\left.+\frac{\sinh(\chi'/2)}{\pi}\mathrm{P}\int\limits_{0}^{\infty} d\chi\frac{\cosh(\chi/2)\ln\left[\pi\varepsilon_{ln}A_{ln}(\chi)/2\right]}{\cosh\chi-\cosh\chi'}\right]\right\},$$
$$n = 1, 2, ..., M_l.$$

Thus, we have shown that, a solution of the relativistic inverse scattering problem exists and is completely determined by the additions of the phase shift and by the energy of true bound state of the total interaction.

In conclusion, we note that method proposed here for reconstructing the nonlocal separable components of the total interaction between two relativistic spinless particles of unequal masses is in fact equivalent to the one-body relativistic inverse scattering problem. This is due to the possibility of representing, within the relativistic quasipotential approach to quantum field theory, the total c.m. energy of two relativistic spinless particles of unequal masses as an expression proportional to the energy of an effective relativistic particle of mass m' [15].

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