Threshold S- and P-Factors in the Relativistic Quasipotential Approach: the Case of Unequal Masses

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Abstract

The new relativistic Coulomb-like threshold S- and P-factors in quantum chromodynamics are presented. Consideration is given within the framework of quasipotential approach in quantum field theory formulated in the relativistic configurational representation in the case of two particles of unequal masses.

1 Introduction

At the description of quark-antiquark systems close to threshold we can not cut off the perturbative series even if the expansion parameter, the QCD coupling constant α_S, is small [1]. The problem is well known from QED [2]. The reason consist in that the real expansion parameter in the threshold region is α/v, where $v = \sqrt{1 - 4m^2/s}$ is a quark velocity, and m is a quark mass. Obviously, it becomes to be singular, when the velocity $v \to 0$. To obtain meaningful result these threshold singularities of the form $(\alpha/v)^n$ have to be summarized. In the nonrelativistic of case for the Coulomb interaction

$$V(r) = -\alpha/r \tag{1}$$

this resummation is realized the known S-factor Gamov–Sommerfeld–Sakharov [3, 4, 5]

$$S_{\rm nr}(v_{\rm nr}) = \frac{X_{\rm nr}(v_{\rm nr})}{1 - \exp\left[-X_{\rm nr}(v_{\rm nr})\right]}, \qquad X_{\rm nr}(v_{\rm nr}) = \frac{\pi\,\alpha}{v_{\rm nr}}, \tag{2}$$

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which is related to the wave function of the continuous spectrum at the origin by $|\psi(0)|^2$. Here $2\,v_{\rm nr}$ is the relative velocity of two nonrelativistic particles. The corresponding nonrelativistic expression can also be obtained for higher ℓ states (see, e.g., [6]).

In the relativistic theory the nonrelativistic approximation needs to be modified. For the first time the relativistic modification of the S-factor (2) in QCD in the case of two particles of equal masses ($m_1 = m_2 = m$) was executed in [7] and it consisted in the change $v_{\rm nr} \to v$. This factor was used for the description of effects close to the threshold of pair production in the processes $e^+e^- \to t\bar{t}$ and $e^+e^- \to W^+W^-$. Just the same form of the S-factor for the interaction of two particles of equal masses was later suggested in [8]. Another form of the relativistic generalization of the S-factor also in the case of two particles of equal masses was obtained in [9]. The relativistic S-factor for two particles of arbitrary masses ($m_1 \neq m_2$) was presented in [10]. This factor was derived within the framework of relativistic quantum mechanics on the basis of the Schrödinger equation.

The new method to relativistic generalization of the S-factor in the case of two particles of equal masses was developed by Milton and Solovtsov in [11]. Their the method is based on the relativistic quasipotential (RQP) approach proposed by Logunov and Tavkhelidze [12] in the form suggested by Kadyshevsky [13]. In the method developed by them, the possibility of transformation of quasipotential (QP) equation from momentum space into relativistic configurational representation in the case of two particles of equal masses (see [14]) has been used also. Moreover, it is important the potential (1) that used by them possesses the QCD-like behaviour (see [15]). The solution containing arbitrary functions of r with period i, the so-called the i-periodic constants, with the same potential was investigated in [16]. However, the using of such solution is suitable for the spectral problems only.

Thus, in [11] a new step to application of the QP approach in QCD was made. This approach gives the following expression for the relativistic S-factor:

$$S(\chi) = \frac{X(\chi)}{1 - \exp\left[-X(\chi)\right]}, \qquad X(\chi) = \frac{\pi\,\alpha}{\sinh\chi}, \tag{3}$$

where χ is the rapidity related to the total c. m. energy of interacting particles, $\sqrt{s}$, by $2m\cosh\chi = \sqrt{s}$. The function $X(\chi)$ in Eq. (3) can be expressed in terms of v as $X(\chi) = \pi\,\alpha\sqrt{1-v^2}/v$. The method proposed by them in [11] has been applied in [17] successfully to get the following expression for the relativistic P-factor (for $\ell = 1$ state) in the case of two particles of equal masses:

$$P(\chi) = \left(1 + \frac{\alpha^2}{4\sinh^2\chi}\right) S(\chi) . \tag{4}$$

In that paper, a new model expression for $R(s)$, in which threshold singularities were summarized to the main potential contribution, was suggested as well. The generalization of the relativistic S- and P-factors for arbitrary ℓ states in the case of two particles of equal masses was executed in [18, 19]. Applications of the factor (3) for describing some hadronic processes can be found in [20, 21, 22]. Recently, the relativistic S-factor (3) has been applied in [23] to reanalyze the mass limits obtained for magnetic monopoles which might have been produced at the Fermilab Tevatron.

We should like to remind that the resummation factors appears in the parametrization of the imaginary part of the quark current correlator, the Drell ratio $R(s)$, which can be approximated in terms of the Bethe-Salpeter (BS) amplitude of two charged particles $\chi_{\mathrm{BS}}(x)$ at $x = 0$ (see [24]). The nonrelativistic replacement of this amplitude by the wave function, which obeys the Schrödinger equation with the Coulomb potential (1), gives the formula (2) with a substitution $\alpha \to 4\,\alpha_{\mathrm{S}}/3$ for QCD. The possibility of using the QP approach for our task is based on the fact that the QP wave function in the momentum space, $\Psi_q(\mathbf{p})$, is defined as the BS amplitude is taken at $x = 0$ by the relation

$$\chi_{\mathrm{BS}}(x = 0) = \frac{1}{(2\pi)^3} \int d\Omega_p \, \Psi_q(\mathbf{p}) \,, \tag{5}$$

where $d\Omega_p = (m\, d\mathbf{p})/E_p$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid $E_p^2 - \mathbf{p}^2 = m^2$.

The purpose of this paper is to generalize the previous study started in [11] to the case of the interaction of two particles of unequal masses ($m_1 \neq m_2$). The method is based on the RQP approach in quantum field theory proposed by Kadyshevsky in [13] formulated in the relativistic configuration representation for the interaction of two relativistic particles of unequal masses [25]. Within the framework of this approach we derive the new relativistic S- and P-factors and analyze their behavior in the following cases: the nonrelativistic and relativistic cases, the case of equal masses, and the ultrarelativistic case. In the following we will use the system of units $c = \hbar = 1$.

2 The integral form of quasipotential equation in the case of two particles of unequal masses

The basis of our consideration is QP equation into the momentum space constructed in [25] for the RQP wave function $\Psi_{q'}(\mathbf{p}')$ of two relativistic particles of unequal masses. This equation is given by

$$(2E_{q'} - 2E_{p'})\, \Psi_{q'}(\mathbf{p}') = \frac{2\,\mu}{m'\,(2\pi)^3} \int d\Omega_{k'}\, \widetilde{V}\,(\mathbf{p}'\,, \mathbf{k}';\, E_{q'})\, \Psi_{q'}(\mathbf{k}')\,, \tag{6}$$

where $d\Omega_{k'} = m'\, d\mathbf{k}'/E_{k'}$ is the relativistic three-dimensional volume element in the Lobachevsky space, $E_{k'} = \sqrt{m'^{\,2} + \mathbf{k}'^{\,2}}, m' = \sqrt{m_1 m_2}$, and $\mu = m_1 m_2/(m_1 + m_2)$ is the usual reduced mass.

Eq. (6) represents a relativistic generalization of the Lippmann-Schwinger equation in the spirit of the Lobachevsky geometry, which is realized on the upper half of the mass hyperboloid $E^2_{k'} - \mathbf{k}'^{\,2} = m'^{\,2}$. This equation describes the scattering over the quasipotential $\widetilde{V}\,(\mathbf{p}',\, \mathbf{k}';\, E_{q'})$ of an effective relativistic particle having mass m' and a relative 3-momentum $\mathbf{k}'$, emerging instead of the system of two particles and carrying the total c. m. energy of the interacting particles, $\sqrt{s}$, proportional to the energy $E_{k'}$ of one effective relativistic particle of mass m' (see [25, 26]):

$$\sqrt{s} = \sqrt{m_1{}^2 + \mathbf{k}^2} + \sqrt{m_2{}^2 + \mathbf{k}^2} = \frac{m'}{\mu} E_{k'}\,. \tag{7}$$

The proper Lorentz transformations means a translation in the Lobachevsky space. The role of the plane waves corresponding to these translations are played by the following functions:

$$\xi(\mathbf{p}'\,, \mathbf{r}) = \left(\frac{E_{p'} - \mathbf{p}' \cdot \mathbf{n}}{m'} \right)^{-1-ir\,m'}\,, \tag{8}$$

where the module of the radius-vector, r, $(\mathbf{r} = r\,\mathbf{n}\,, |\mathbf{n}| = 1)$ is a relativistic invariant [26]. The functions (8) correspond to the principal series of unitary representations of the Lorentz group and they obey the conditions of completeness and orthogonality (see [26]).

We note that Eq. (6) differs from of the QP equation considered in [27] by means of introduction into it of the relativistic reduced mass. However, in [27] was shown that it is possible to use the different expressions for the relativistic reduced mass by means of the choice of functional relationship between the relative 3-momentum $\mathbf{k}$ and the relativistic relative velocity of interacting particles, v, connected with their the total energy, $\sqrt{s}$, by relation (see, for instance, [9, 10])

$$\mathrm{v} = 2\sqrt{\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}}\left(1+\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}\right)^{-1} . \tag{9}$$

In particular, if the dependence between the energy of the relative motion and the relativistic relative velocity v is given by expression (see [25, 26])

$$\frac{\mathbf{k}'^2}{2\mu} = \mu\left(\frac{1}{\sqrt{1-\mathrm{v}^2}}-1\right) , \tag{10}$$

this together with relation (9) gives the expression (7). Such the choice of functional relationship has allowed to enter the concept of an effective relativistic particle [25, 26]. Notice that the relative 3-momentum $\mathbf{k}'$ of an effective relativistic particle, according to the expression (10), is invariant of the Loretz transformations.

For a spherically symmetric potential the application of Shapiro transformations (or ξ-transformations [25, 26])

$$\begin{aligned} \psi_{q'}(\mathbf{r}) &= \frac{1}{(2\pi)^3}\int d\Omega_{p'}\, \xi(\mathbf{p}'\,,\mathbf{r})\, \Psi_{q'}(\mathbf{p}')\,, \\ \Psi_{q'}(\mathbf{p}') &= \int d\mathbf{r}\, \xi^*(\mathbf{p}'\,,\mathbf{r})\, \psi_{q'}(\mathbf{r})\,, \end{aligned} \tag{11}$$

to Eq. (6) it has lead us to the equation, which is the integral form of the relativistic Schrödinger equation in the configurational representation:

$$\begin{aligned} \frac{1}{(2\pi)^3}\int d\Omega_p\, (2E_q-2E_p)\, \xi(\mathbf{p}\,,\boldsymbol{\rho})\int d\boldsymbol{\rho}'\, \xi^*(\mathbf{p}\,,\boldsymbol{\rho}')\, \psi_q(\boldsymbol{\rho}') = \\ = \frac{2\,\mu}{m'}\, V(\rho\,;E_q)\, \psi_q(\boldsymbol{\rho})\,. \end{aligned} \tag{12}$$

Here the right-hand side is already local in the configuration representation, the transform of the potential, $V(\rho\,;E_{q'})$, is given in terms of the same relativistic plane waves and where we introduced the following notation:

$$\mathbf{q}' = m'\mathbf{q}\,,\mathbf{p}' = m'\mathbf{p}\,,\mathbf{q} = \sinh(\chi_q)\,\mathbf{n}_q\,,\mathbf{p} = \sinh(\chi_p)\,\mathbf{n}_p\,,$$
$$|\mathbf{n}_q| = |\mathbf{n}_p| = 1\,,\boldsymbol{\rho} = m'\mathbf{r}\,,\boldsymbol{\rho}' = m'\mathbf{r}'\,,\rho = |\boldsymbol{\rho}|\,,d\mathbf{r}' = m'^{-3}d\boldsymbol{\rho}'\,,$$
$$d\Omega_{p'} = m'^3 d\Omega_p\,,d\Omega_p = \frac{d\mathbf{p}}{E_p}\,,E_{q'} = m'E_q\,,E_{p'} = m'E_p\,, \tag{13}$$
$$E_q = \sqrt{1+\mathbf{q}^2}\,,E_p = \sqrt{1+\mathbf{p}^2}\,,V(r\,;E_{q'}) \equiv m'V(\rho\,;E_q)\,,$$
$$\xi(\mathbf{p}'\,,\mathbf{r}) \equiv \xi(\mathbf{p}\,,\boldsymbol{\rho})\,,\psi_{q'}(\mathbf{r}) \equiv \psi_q(\boldsymbol{\rho})\,,\Psi_{q'}(\mathbf{p}') \equiv m'^{-3}\Psi_q(\mathbf{p})\,.$$

By using the expansions

$$\xi(\mathbf{p}\,,\boldsymbol{\rho}) = \sum_{\ell=0}^{\infty}(2\ell+1)\,i^{\ell}\,p_{\ell}(\rho\,,\cosh\chi_p)\,P_{\ell}\left(\frac{\mathbf{p}\cdot\boldsymbol{\rho}}{p\,\rho}\right)\,,$$
$$\psi_q(\boldsymbol{\rho}) = \sum_{\ell=0}^{\infty}(2\ell+1)\,i^{\ell}\,\frac{\varphi_{\ell}(\rho\,,\chi_q)}{\rho}\,P_{\ell}\left(\frac{\mathbf{q}\cdot\boldsymbol{\rho}}{q\,\rho}\right)\,, \tag{14}$$

and also formula [14] $p_{\ell}(\rho\,,\cosh\chi) = \dfrac{(-1)^{\ell}\,(\sinh\chi)^{\ell}}{\rho^{(\ell+1)}}\left(\dfrac{d}{d\cosh\chi}\right)^{\ell}\left(\dfrac{\sin\rho\chi}{\sinh\chi}\right)$, Eq. (12) transformed to the form

$$\frac{2}{\pi}\int_0^{\infty}d\chi'\frac{(\sinh\chi')^{2\ell+2}\,(-1)^{\ell+1}}{\rho^{(\ell+1)}}\,(2\cosh\chi - 2\cosh\chi')\left(\frac{d}{d\cosh\chi'}\right)^{\ell}\times$$
$$\times\left(\frac{\sin\rho\chi'}{\sinh\chi'}\right)\left(\frac{d}{d\cosh\chi'}\right)^{\ell}\frac{1}{\sinh\chi'}\int_0^{\infty}d\rho'\frac{\rho'\,\sin\rho'\chi'}{(-\rho')^{(\ell+1)}}\,\varphi_{\ell}(\rho'\,,\chi) = \tag{15}$$
$$= \frac{2\,\mu}{m'}\,\frac{V(\rho\,;E_q)\,\varphi_{\ell}(\rho\,,\chi)}{\rho}\,.$$

Here

$$(-\rho)^{(\ell+1)} = i^{\ell+1}\frac{\Gamma(\ell+1+i\rho)}{\Gamma(i\rho)} \tag{16}$$

is the generalized power [14], $\Gamma(z)$ is the gamma-function, $P^{\nu}_{\mu}(z)$ is a Legendre function of the first kind, and the function $p_{\ell}(\rho\,,\cosh\chi)$ is the solution of Eq. (12) in the case when the interaction is switched off, $V(\rho\,;E_q) \equiv 0$; χ' and χ are the rapidities which are related to E_p and E_q as $E_p = \cosh\chi'\,,E_q = \cosh\chi$.

Thus, Eq. (15) differs from the corresponding equation in the case of two particles of equal masses (see [19]) only by the factor $2\mu/m'$ turning into 1 at $m_1 = m_2$.

3 Relativistic threshold S- and P-factors

We note that the applying of ξ-transformation (11) to the Coulomb interaction (1) gives the potential in momentum space $V(\Delta) \sim (\chi_\Delta \sinh \chi_\Delta)^{-1}$, where the relative rapidity χ_Δ corresponds to $\mathbf{\Delta} = \mathbf{p}'(-)\mathbf{k}'$ and is defined in terms of the square of the momentum transfer by $Q^2 = -(p' - k')^2 = 2(\cosh \chi_\Delta - 1)$. For large Q^2 the potential $V(\Delta)$ behaves as $(Q^2 \ln Q^2)^{-1}$, which reproduces the principal behaviour of the QCD potential proportional to $\bar{\alpha}_S(Q^2)/Q^2$ with $\bar{\alpha}_S(Q^2)$ being the QCD running coupling. This property of the potential (1), its QCD-like behaviour, was noted in [15].

We will seek a solution of RQP equation (15) with the potential (1) in the form (see [11, 28, 19])

$$\varphi_\ell(\rho\,,\chi) = \frac{(-\rho)^{(\ell+1)}}{\rho} \int\limits_{\alpha_-}^{\alpha_+} d\zeta\, e^{i\rho\zeta}\, R_\ell(\zeta\,,\chi)\,, \tag{17}$$

where the ζ-integration is performed in the complex plane over a contour with end points α_- and α_+ (see Fig. 1). Substituting (17) into (15) we arrive at the equation

$$\begin{aligned}(-1)^\ell \int\limits_{\alpha_-}^{\alpha_+} d\zeta\, R_\ell(\zeta\,,\chi)\left(\frac{d}{d\cosh\zeta}\right)^\ell \Big[(\sinh\zeta)^{2\ell+1}(2\cosh\chi - \\ -2\cosh\zeta)\left(\frac{d}{d\cosh\zeta}\right)^\ell \left(\frac{e^{i\rho\zeta}}{\sinh\zeta}\right)\Big] = \\ = -\frac{2\,\alpha\,\mu}{m'\,\rho}\prod_{n=1}^{\ell}(\rho^2+n^2)\int\limits_{\alpha_-}^{\alpha_+} d\zeta\, e^{i\rho\zeta}\, R_\ell(\zeta\,,\chi)\,.\end{aligned} \tag{18}$$

Eq. (18) at $\ell = 0$, when we integrate it by parts, has lead us to the equation

$$\frac{d}{d\zeta}\Big[(\cosh\chi - \cosh\zeta)\, R_0(\zeta\,,\chi)\Big] - \frac{i\alpha\,\mu}{m'}\, R_0(\zeta\,,\chi) = 0 \tag{19}$$

with the boundary condition

$$e^{i\rho\zeta}\,(\cosh\chi - \cosh\zeta)\, R_0(\zeta\,,\chi)\Big|_{\zeta=\alpha_-}^{\zeta=\alpha_+} = 0\,. \tag{20}$$

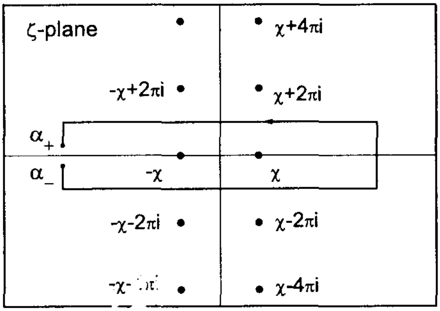

Figure 1. Contour of integration in Eq. (17) and singularities of the function (21) in the complex ζ-plane

As a result the solution of Eq. (19) is

$$R_0(\zeta, \chi) = C_0(\chi) \frac{e^{\zeta}}{(e^{\zeta} - e^{\chi})^2} \left[\frac{e^{\zeta} - e^{-\chi}}{e^{\zeta} - e^{\chi}} \right]^{-1+iA}, \quad A = \frac{\alpha \mu}{m' \sinh \chi}, \tag{21}$$

where $C_0(\chi)$ is an arbitrary function of χ. The branch points of the function (21) are $\pm\chi + 2\pi n i$, $n = 0, \pm 1, \ldots$ (see Fig. 1). The contour of integration must not intersect cuts which we take from $-\infty + 2\pi n i$ to $\pm\chi + 2\pi n i$. In the case when the interaction vanishes, $\alpha \to 0$, the solution $\varphi_\ell(\rho, \chi)$ should reproduce the known free wave function $\rho\, p_\ell(\rho, \cosh\chi)$:

$$\lim_{\alpha \to 0} \varphi_\ell(\rho, \chi) = \rho\, p_\ell(\rho, \cosh\chi) \xrightarrow[\rho\to\infty]{} \frac{\sin(\rho\chi - \pi\ell/2)}{\sinh\chi}. \tag{22}$$

Taking into account these remarks and the boundary condition (20), we take: $\alpha_- = -R - i\varepsilon, \alpha_+ = -R + i\varepsilon, \mathrm{Re}\,\zeta = +R, \mathrm{Im}\,\zeta = \pm\pi$ with $R \to +\infty, \varepsilon \to +0$ (Fig. 1). It is also convenient for finding a connection to an integral representation of the hypergeometric function. Substituting the solution (21) into (17) at $\ell = 0$ and performing ζ-integration in the complex plane along a contour with end points α_- and α_+ (in the same way as in [11, 28, 19]) we obtain the resulting solution for the RQP partial wave function $\varphi_0(\rho, \chi)$ in the form

$$\varphi_0(\rho, \chi) = C_0(\chi) \frac{2\rho \sinh(\pi\rho)}{\rho^{(1)}} \int_{-\infty}^{\infty} dx \frac{e^{(i\rho+1)x}}{(e^{x} + e^{\chi})^2} \left[\frac{e^{x} + e^{-\chi}}{e^{x} + e^{\chi}} \right]^{-1+iA}, \tag{23}$$

where the function $\rho^{(1)}$ is determined in Eq. (16). The solution (23) does not contain the i-periodic constant and can also be represented in terms of hypergeometrical function as

$$\varphi_0(\rho\,,\chi) = -N_0(\chi)(-\rho)^{(1)}\, e^{i\rho\chi+iA\chi}\, F\left(1-iA, 1-i\rho; 2; 1-e^{-2\chi}\right)\,. \quad (24)$$

The normalization constant $N_0(\chi)$ in Eq. (24) can be obtained from the condition (22) at $\ell = 0$.

The BS amplitude $\chi_{\rm BS}(x=0)$ is associated with the RQP wave function in the momentum space, $\Psi_q(\mathbf{p})$, by the relation (5). Taking into account the transformations (11) and notations (13), the relationship of the BS amplitude with the RQP wave function, $\psi_q(\boldsymbol{\rho})$, is $\chi_{\rm BS}(x=0) = \psi_q(\boldsymbol{\rho})|_{\rho=i}$. The generalized power (16) in the solution (17) vanishes at $\rho = i$ for all $\ell \neq 0$. Thus, the expansion (14) for the wave function $\psi_q(\boldsymbol{\rho})$ contains only s-wave ($\ell = 0$). Hence, by using relations (24) and (22) at $\ell = 0$ we can calculate $|\psi_q(i)|^2$, which has lead us to the following expression for the relativistic S-factor in the case of two particles of unequal masses:

$$\begin{aligned} S_{\rm uneq}(\chi) &= \lim_{\rho\to i}\left|\frac{\varphi_0(\rho\,,\chi)}{\rho}\right|^2 = \frac{X_{\rm uneq}(\chi)}{1-\exp\left[-X_{\rm uneq}(\chi)\right]}\,, \\ X_{\rm uneq}(\chi) &= \frac{2\,\pi\,\alpha\,\mu}{m'\,\sinh\chi}\,, \end{aligned} \quad (25)$$

where χ is the rapidity which is related to the $\sqrt{s}$ as $\dfrac{m'^2}{\mu}\cosh\chi = \sqrt{s}$.

Eq. (18) at $\ell = 1$, when we integrate it by parts, has lead us to the equation

$$\begin{aligned} &\frac{d}{d\zeta}\left\{\frac{1}{\sinh\zeta}\frac{d}{d\zeta}\left[(\sinh\zeta)^2\,(\cosh\chi-\cosh\zeta)\,\frac{d}{d\zeta}\left(\frac{R_1(\zeta\,,\chi)}{\sinh\zeta}\right)\right]\right\}- \\ &-\frac{i\alpha\,\mu}{m'}\left[\frac{d^2}{d\zeta^2}\Big(R_1(\zeta\,,\chi)\Big) - R_1(\zeta\,,\chi)\right] = 0 \end{aligned} \quad (26)$$

with the boundary conditions

$$
\begin{aligned}
&e^{i\rho\zeta}\left(\cosh\chi-\cosh\zeta\right)R_1(\zeta\,,\chi)\Big|_{\zeta=\alpha_-}^{\zeta=\alpha_+}=0\,,\\
&-e^{i\rho\zeta}\Big\{\left(\cosh\chi-\cosh\zeta\right)\Big[\frac{\cosh\zeta}{\sinh\zeta}R_1(\zeta\,,\chi)+\\
&+\sinh\zeta\frac{d}{d\zeta}\Big(\frac{R_1(\zeta\,,\chi)}{\sinh\zeta}\Big)\Big]-\frac{i\alpha\,\mu}{m'}R_1(\zeta\,,\chi)\Big\}\Big|_{\zeta=\alpha_-}^{\zeta=\alpha_+}=0\,, \qquad (27)\\
&e^{i\rho\zeta}\Big\{\frac{1}{\sinh\zeta}\frac{d}{d\zeta}\Big[(\sinh\zeta)^2\left(\cosh\chi-\cosh\zeta\right)\frac{d}{d\zeta}\Big(\frac{R_1(\zeta\,,\chi)}{\sinh\zeta}\Big)\Big]-\\
&-\frac{i\alpha\,\mu}{m'}\frac{d}{d\zeta}\Big(R_1(\zeta\,,\chi)\Big)\Big\}\Big|_{\zeta=\alpha_-}^{\zeta=\alpha_+}=0\,.
\end{aligned}
$$

Substituting the solution of Eq. (26) into (17) at $\ell\;=\;1$ and performing ζ-integration in the complex plane along a contour with end points α_- and α_+ (in the same way as in case s-wave) we obtain the resulting solution for the RQP partial wave function $\varphi_1(\rho\,,\chi)$ in the form

$$
\varphi_1(\rho,\chi)=-C_1(\chi)\frac{2\rho\,\sinh(\pi\rho)}{\rho^{(2)}}\int\limits_{-\infty}^{\infty}dx\frac{e^{(i\rho+2)x}}{(e^x+e^\chi)^4}\left[\frac{e^x+e^{-\chi}}{e^x+e^\chi}\right]^{-2+iA}\,, \qquad (28)
$$

where the function $\rho^{(2)}$ is determined in (16). The solution (28) does not contain the i-periodic constant and can also be represented in terms of hypergeometrical function as

$$
\varphi_1(\rho\,,\chi)=N_1(\chi)(-\rho)^{(2)}\,e^{i\rho\chi+iA\chi}\,F\left(2-iA\,,2-i\rho\,;4\,;1-e^{-2\chi}\right)\,. \qquad (29)
$$

The normalization constant $N_1(\chi)$ in Eq. (29) can be obtained (also as in case s-wave) from the condition (22) at $\ell=1$. By using Eq. (29) and the condition (22) at $\ell\;=\;1$, we find the following expression for the relativistic P-factor, which corresponds p-wave (see [17]), in the case of two particles of unequal masses:

$$
P_{\rm uneq}(\chi)=\lim_{\rho\to i}\left|\frac{3}{\sinh\chi}\Delta^*\left(\frac{\varphi_1(\rho\,,\chi)}{\rho}\right)\right|^2=\left(1+A^2\right)\,S_{\rm uneq}(\chi)\,, \qquad (30)
$$

where $\Delta^*=\dfrac{1}{i}\left[\exp\left(i\dfrac{\partial}{\partial\rho}\right)-1\right]$ is finite difference derivative [14, 26].

The function $X_{\text{uneq}}(\chi)$ and parameter A in Eqs. (21), (25) and (30) can be expressed in terms of the "velocity" u determined by the relation

$$u = \sqrt{1 - \frac{4\,m'^2}{\bar{s}}}\,, \qquad \bar{s} = s - (m_1 - m_2)^2\,, \tag{31}$$

in the form

$$X_{\text{uneq}}(\chi) = \frac{\pi\,\alpha\,\sqrt{1-u^2}}{u}\,, \qquad A = \frac{\alpha\,\sqrt{1-u^2}}{2u}\,. \tag{32}$$

The square of relative 3-momentum $\mathbf{k}'$ for an effective relativistic particle, emerging instead of the system of two particles, is connected with the relative relativistic velocity of interacting particles, v, by the expression (10). Thence, taking into consideration the determination (31) and relation (9), we find

$$\text{v} = \frac{2\,u}{1+u^2}\,, \qquad \mathbf{k}'^2 = (\mu'_{\text{rel}})^2\,(u'_{\text{rel}})^2\,, \tag{33}$$

where $\mu'_{\text{rel}} = 2\,\mu$ is the relativistic reduced mass, and

$$u'_{\text{rel}} = \frac{u}{\sqrt{1-u^2}} \tag{34}$$

is the relative velocity of an effective relativistic particle of mass m'. This result is found to be in full agreement with the physical meaning of Eq. (6).

Thus, in terms of relative velocity of an effective relativistic particle (34), the S-factor (25) and P-factor (30) are given by expressions

$$S_{\text{uneq}}(u'_{\text{rel}}) = \frac{X_{\text{uneq}}(u'_{\text{rel}})}{1 - \exp\left[-X_{\text{uneq}}(u'_{\text{rel}})\right]}\,, \quad X_{\text{uneq}}(u'_{\text{rel}}) = \frac{\pi\,\alpha}{u'_{\text{rel}}}\,; \tag{35}$$

$$P_{\text{uneq}}(u'_{\text{rel}}) = \left[1 + \left(\frac{\alpha}{2\,u'_{\text{rel}}}\right)^2\right] S_{\text{uneq}}(u'_{\text{rel}})\,. \tag{36}$$

The factors in Eqs. (35) and (36) only formally have the same forms, as their the nonrelativistic analogies. However, these the factors have an obviously relativistic nature since as the argument $r = |\mathbf{r}|$ in the Coulomb potential (1) and the relativistic relative velocity of interacting particles, v, (see [26]) both are relativistic invariants. Hence the relative velocity of an effective relativistic particle (34), according to Eqs. (10) and (33), possesses this property as well.

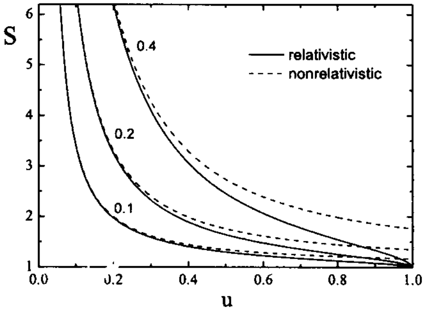

Figure 2. Behavior of the S-factor at different values of α (the numbers at the curves).

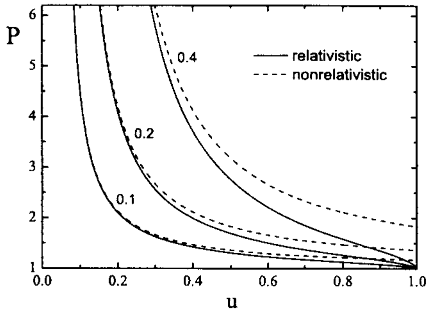

Figure 3. Behavior of the P-factor at different values of α (the numbers at the curves).

The relativistic threshold factors (35) and (36) have the following important properties:

• In the nonrelativistic limit, $u \ll 1$, they reproduces the well-known nonrelativistic results.

• In the relativistic limit, $u \to 1$, both factors (35) and (36) go to unity.

• In the case of equal masses they coincides with S-factor (3) and P-factor (4).

• In the ultrarelativistic limit, as it was argued in [29], the bound state spectrum vanishes since mass of an effective relativistic particle $m' \to 0$. This feature reflects an essential difference between potential models and quantum field theory where an additional dimensional parameter appears. One can conclude that within the framework of a potential model, the S- and P-factors which correspond to the continuous spectrum should go to unity in the limit $m' \to 0$. Thus, in contrast to the nonrelativistic case, the relativistic threshold the S- and P-factors in Eqs. (35) and (36), reproduces both the known nonrelativistic and the expected ultrarelativistic limits.

To illustrate differences between the new relativistic S- and P-factors in Eqs. (35) and (36) and their nonrelativistic analogies in more detail, in Figs. 2 and 3 we plot the behavior of these factors as functions of u at different values of the parameter α (the numbers at the curves). The solid lines correspond to the relativistic S- and P-factors in Eqs. (35) and (36); the dashed lines to the nonrelativistic S-factor (2) and P-factor (see [6])

$$P_{\rm nr}(v_{\rm nr}) = \left[1 + \left(\frac{\alpha}{2\,v_{\rm nr}}\right)^2\right] S_{\rm nr}(v_{\rm nr})$$

with a substitution $v_{\rm nr} \to u$. From these figures one can see that in the region of nonrelativistic values of u, $u \leq 0.2$, where the influence of the S- and P-factors are big, the differences between (35) and (36) and their nonrelativistic analogies are practically absent. However, when α increases, the nonrelativistic expressions gives a less suitable result in the region of large values u, in particular, as $u \to 1$.

4 Conclusion

The new relativistic threshold S- and P-factors [(35) and (36)] for the interaction of two relativistic particles of unequal masses were obtained. For this aim the RQP equation in relativistic configuration representation [25] with the Coulomb potential for the interaction of two relativistic particles of unequal masses was used. The new relativistic threshold factors obtained here reproduce both the known nonrelativistic and expected ultrarelativistic limits and correspond to the QCD-like Coulomb potential. The Coulomb potential only formally has the same form as the nonrelativistic potential but differs in the relativistic configuration representation since its behavior corresponds to the quark-antiquark potential $V_{q\bar{q}} \sim \bar{\alpha}_s(Q^2)/Q^2$ with the invariant charge $\bar{\alpha}_s(Q^2) \sim 1/\ln Q^2$. So, the principal effect coming from the running of the QCD coupling is accumulated.

The new S- and P- factors are coincided in form with their nonrelativistic analogies. However, the role of the parameter of velocity is played not by the relativistic relative velocity of interacting particles, v, but by the relative velocity (34) of an effective relativistic particle emerging instead of the system of two particles. It was shown that there are a differences between the expressions (35) and (36) the obtained here and their nonrelativistic analogies. As the new relativistic threshold factors (35) and (36) were obtained within the framework of completely covariant method, one can expect that these factors takes into account more adequately relativistic nature of interaction.

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