Analytic properties and renormalization scheme dependence in variational perturbation theory

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Abstract

It is shown that applying the nonperturbative method of variational perturbation series allows us to considerably reduce the renormalization scheme dependence in QCD calculations.

1 Introduction

Perturbation theory (PT) is commonly used in theoretical calculations in QCD. In practices one usually applies the popular $\overline{\mathrm{MS}}$ renormalization scheme (RS). Physical quantities are independent of the particular choose of the RS. However, in real calculations this dependence is appeared due to a inevitable truncation of perturbation series. The RS-dependence is a source of essential theoretical uncertainties which become to be large especially at low energy scale. At low energy experimental data are sensitive to important nonperturbative characteristics and, therefore, theoretical uncertainties disturb to investigate nonperturbative effects accurately. There are no general principles allowing us to decide which scheme is preferable and it is necessary to consider the stability results obtained with respect to the choose of RS.

Another way is a modification of the perturbative component of the QCD calculations. The point is that the initial PT series, or more precisely, its finite part after renormalization, is not the final product of the theory but admits of a considerable modification. In particular, it is well known that the renormalization group method [1] allows one to modify a perturbative expansion in accordance with the general principle of renormalization invariance, thus improving the properties of the series in the ultraviolet region. As to the infrared region, where the perturbative invariant charge possesses unphysical singularities, the RG-modified PT series remains unstable. In the analytic approach in QCD proposed by D.V. Shirkov and coauthor of this paper in [2] the trouble with unphysical singularities has found its possible resolution. It has been demonstrated that the framework of the analytic approach the RS-dependence can be essentially reduced [3, 4, 5, 6]. Others advantages of analytic perturbation theory has been recently emphasized in [9, 12, 11, 10].

Here to investigate the RS-dependence we use the nonperturbative method of constructing the so-called floating or variational series in quantum chromodynamics suggested in [7, 8].

2 Variational perturbation theory and analyticity

In the usual version of perturbation theory, the total action corresponding to a physical system is split into a free part and a part describing the interaction. The latter is treated as a perturbation, and the coupling constant entering into it is viewed as the small expansion parameter. As a rule, this treatment leads to asymptotic series which, albeit not "well behaved," nevertheless widely used in physics and allow useful information about the system in question to be extracted for weak coupling. As the interaction constant grows, the perturbation theory becomes worse and worse. The reason for this is understood: now the treatment of the interaction term as a perturbation of the free system is no longer adequate, since the physical system in question has properties far from those of a free system. In order to have a method of performing calculations in this case, it is necessary to split the total action in different way, such that the new "interaction term" can be treated as a perturbation not only when the coupling constant is small, but for a wider range of its value. Of course, here one must worry about whether this procedure, which is similar to ordinary perturbation theory, allows the possibility of calculating a main contribution and corrections.

How is it possible to seek a functional which can be used as a perturbation with more justification than the usual interaction term? The method proposed in [7, 8] based on the idea of variational perturbation theory

(VPT), which in the case of QCD leads to a new small expansion parameter [7, 8] (see also reviews [13, 14]). Within this method, a quantity under consideration can be represented in the form of a series, which is different from the conventional perturbative expansion and can be used to go beyond the weak-coupling regime. This allows one to deal with considerably lower energies than in the case of perturbation theory [15, 16, 17].

Analysis of the structure of this variational perturbation series shows that it can be organized in powers of the new small parameter a if the standard coupling constant g is related to a by

$$\lambda = \frac{1}{C}\frac{a^2}{(1-a)^3}\,, \tag{1}$$

where $\lambda = g^2/(4\pi)^2 = \alpha_s/4\pi$ and C is a positive constant. As follows from (1), at any values of the coupling constant g, the new expansion parameter a obeys the inequality

$$0 \leq a < 1\,. \tag{2}$$

The positive parameter C plays the role of an auxiliary parameter of a variational type,which is associated with the use of a floating series. The original quantity which is approximated by this expansion does not depend on the auxiliary parameters C; however, any finite approximation depends on it on account of the truncation of the series.

The variational parameter C can be defined, if one takes into account, as in the Shirkov–Solovtsov analytic approach, the Källén–Lehmann analyticity. As it has been demonstrated in [18, 19] its value is changing from order to order, in accordance with the phenomenon of induced convergence.[1] Taking into account the order $O(a^k)$ for C_k we have

$$C_3 = 3.5, \quad C_4 = 9.2, \quad C_5 = 19.1, \quad C_6 = 34.1, \quad \text{and} \quad C_7 = 55.6\,. \tag{3}$$

The increase of C_k with the oder to of the expansion is explained by the necessity to compensate for the higher order contributions.

It is interesting that these values agree well with values of the parameter coming from the meson spectroscopy [8]. The parameter C can be defined from the condition that the renormalization group β-function at large enough values of the coupling constant behaves as $\beta(\lambda) \simeq -\lambda$. Such

[1]It has been observed empirically in [25, 26] that the results seem to converge if the variational parameter is chosen, in each order, according to some variational principle. This induced-convergence mechanism is also discussed in [27].

a behaviour corresponds to the singular infrared behavior of the running coupling constant $\lambda(Q^2) \sim Q^{-2}$ and leads to the linear growth of the non-relativistic static quark-antiquark potential at large distances $V(r) \sim r$.

3 Renormalization scheme dependence

In QCD it is important to determine "simplest" objects which allow one to check direct consequences of the theory without using model assumptions in an essential manner. Comparison of theoretical results for these objects with experimental data allows us to justify transparently the validity of basic statements of the theory, and make some conclusions about completeness and efficiency of the theoretical methods used. Some single-argument functions which have a straightforward connection with experimentally measured quantities can play the role of these objects. A theoretical description of inclusive processes can be expressed in terms of functions of this sort. Let us mention among them moments $\mathrm{M}_n(Q^2)$ of the structure functions in inelastic lepton-hadron scattering and the hadronic correlator $\Pi(s)$ (or the corresponding Adler D-function), which appear in the processes of e^+e^- annihilation into hadrons or the inclusive decay of the τ lepton.

Consider here as an example the D-function

$$D(Q^2) = -Q^2 \frac{d\Pi}{dQ^2} = Q^2 \int_0^\infty ds \frac{R(s)}{(s+Q^2)^2} \, . \tag{4}$$

Separating the QCD correction $d(Q^2)$ we represent $D(Q^2)$ in the form

$$D(Q^2) \;=\; 3 \sum_f Q_f^2 \left[1 + d_f(Q^2)\right], \tag{5}$$

where Q_f denotes the electric charge of the quark with the flavor f. The expression for $d(Q^2)$ with the running coupling has the form:

$$d_{PT}(Q^2) = \delta_i(Q^2) [\, 1 + d_1^i \delta_i(Q^2) + d_2^i \delta_i^2(Q^2) + \ldots\,] \, , \tag{6}$$

where $\delta_i = \alpha_s^i/\pi$ and the index i denotes the RS in which one performs calculations.

In $\overline{\mathrm{MS}}$-scheme the perturbative coefficients are [20, 21]

$$d_1^{\overline{\mathrm{MS}}} \;=\; 1.986 - 0.115 f \, , \tag{7}$$

$$d_2^{\overline{\mathrm{MS}}} \;=\; 18.244 - 4.216 f + 0.086 f^2 - \frac{1.2395}{3} \frac{(\sum_{f'}^{f} Q_{f'})^2}{\sum_{f'}^{f} Q_{f'}^2} \, . \tag{8}$$

The invariant charge is determined as a solution of the renormalization group equation with the three-loop β-function [22]

$$\beta(\delta) = -\frac{b}{2}\,\delta^2(1 + b_1\delta + b_2\delta^2)\,, \tag{9}$$

where

$$b = \frac{\beta_0}{2}\,, \qquad b_1 = \frac{4\beta_1}{\beta_0}\,, \qquad b_2^{\overline{\rm MS}} = 16\,\frac{\beta_2^{\overline{\rm MS}}}{\beta_0}\,,$$

and

$$\beta_0 = 11 - \frac{2}{3}f\,, \quad \beta_1 = 102 - \frac{38}{3}f\,, \quad \beta_2^{\overline{\rm MS}} = \frac{2857}{2} - \frac{5033}{18}f + \frac{325}{54}f^2\,.$$

In passing from one renormalization scheme to another, RS $\to$ RS$'$, the coupling constant transforms as follows

$$\delta \to \delta' \;=\; \delta(1 + v_1\delta + v_2\delta^2 + \ldots) \tag{10}$$

and the coefficients $d_k \to d'_k$.

In the three-loop level the QCD correction is

$$d \;=\; \delta(1 + d_1\delta + d_2\delta^2)\,. \tag{11}$$

A change in the RS modifies the values of the expansion coefficients. The coefficients b and b_1 are RS independent in the class of mass and gauge independent schemes and the three-loop β-function coefficient b_2 and the expansion coefficients d_1 and d_2 in (6) depend on the choice of the renormalization scheme. Under the scheme transformation (10), they are changing in the following way

$$\begin{aligned} b'_2 &= b_2 - v_1^2 - b_1v_1 + v_2\,,\\ d'_1 &= d_1 - v_1\,,\\ d'_2 &= d_2 - 2(d_1 - v_1)v_1 - v_2\,. \end{aligned} \tag{12}$$

Thus, each term in representation (6) undergoes a transformation, and we thus obtain the new function

$$d' = \delta'(1 + d'_1\delta' + d'_2\delta'^2)\,. \tag{13}$$

where the coupling δ' is evaluated with the new β-function, with the three-loop coefficient b_2 replaced by the primed one b_2'.

There exist two RS invariants [23]

$$\rho_1 = \frac{b}{2}\ln\frac{Q^2}{\Lambda^2} - d_1\,, \qquad \rho_2 = b_2 + d_2 - b_1 d_1 - d_1^2\,. \tag{14}$$

To obtain the invariant charge in arbitrary scheme we use the equation

$$\frac{b}{2}\ln\left(\frac{Q^2}{\Lambda^2_{\overline{\rm MS}}}\right) = d_1^{\overline{\rm MS}} - d_1 + \frac{1}{\delta} + b_1\ln\frac{b\delta/2}{1+b_1\delta} + \Phi(\delta, b_2)\,, \tag{15}$$

where

$$\Phi(\delta, b_2) = b_2\int_0^{\delta}\frac{dx}{(1+b_1x)(1+b_1x+b_2x^2)}\,. \tag{16}$$

Although there are no general arguments to prefer a certain renormalization scheme from the start, we nevertheless can define a class of "natural" schemes, which look reasonable at the three-loop level that we consider. A condition for selecting a class of acceptable schemes has been proposed in [24]. One should restrict oneself to the schemes, where the cancellations between different terms in the second scheme invariant (14) are not too large. Quantitatively, this criterion can be related to the cancellation index

$$C = \frac{1}{|\rho_2|}\left(|b_2| + |d_2| + d_1^2 + |d_1| b_1\right)\,. \tag{17}$$

For the optimal RS based on the principle of minimal sensitivity (PMS) [23, 27] the value of cancellation index is $C_{PMS} \simeq 2$. We will use this value as a boundary for the sufficiently narrow class of natural schemes taking

$$C_{\max} = 2\,. \tag{18}$$

4 RS dependence in VPT

Consider the problem of the RS dependence within the VPT method and compare results obtained with the PT. First of all note, the value of parameter is not too sensitive to the RS-dependent three-loop β-function coefficient b_2.

The third-order VPT β-function in terms of the parameter a has the form

$$\beta = -\frac{1}{C^2}\frac{2\beta_0 a^4}{(2+a)(1-a)^2}(1 + k_1 a + k_2 a^2 + k_3 a^3 + k_4 a^4 + k_5 a^5), \quad (19)$$

where

$$k_1 = \frac{9}{2}, \qquad k_2 = 12 + \frac{\beta_1}{\beta_0 C}, \qquad k_3 = 25 + \frac{15}{2}\frac{\beta_1}{\beta_0 C}, \quad (20)$$
$$k_4 = 45 + \frac{63}{2}\frac{\beta_1}{\beta_0 C} + \frac{\beta_2}{\beta_0 C^2}, \qquad k_5 = \frac{147}{2} + 98\frac{\beta_1}{\beta_0 C} + \frac{21}{2}\frac{\beta_2}{\beta_0 C^2}.$$

Thus, in a new scheme we have

$$d' = \frac{4a^2}{C}\left[1 + 3\,a + \frac{2}{C}(2d_1 + 3C)\,a^2 + \frac{2}{C}(12d_1 + 5C)\,a^3 \right. \quad (21)$$
$$\left. + \frac{84d_1 C + 15C^2 + 16d_2}{C^2}\,a^4 + \frac{224d_1 C + 144d_2 + 21C^2}{C^2}\,a^5\right],$$

where we take $C = C_7 = 55.59$ and function $\Phi(\delta, b_2)$ in (15) is now determined as

$$\Phi_{VPT}(a, b_2) = \frac{\beta_0}{4}\int\limits_0^a \left[\frac{1}{\beta_{VPT}(x)} - \frac{1}{\beta_{PT}^{2-loop}(x)}\right] dx. \quad (22)$$

Consider the Drell ratio $R(s)$ which is defined as the ratio of total cross-sections

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \quad (23)$$

and related to the D-function as follows

$$D(Q^2) = Q^2\int_0^\infty \frac{ds}{(s+Q^2)^2}R(s). \quad (24)$$

Separating in $R(s)$ the QCD correction $r(Q^2)$ as for the D-function in Eq. (5) one can write down the relations between the correction to the Euclidean quantity, $d(Q^2)$, and Minkowskian one, $r(s)$,

$$d(Q^2) = Q^2\int_0^\infty \frac{ds}{(s+Q^2)^2}\,r(s), \quad (25)$$

and

$$r(s) = -\frac{1}{2\pi i} \int_{s-i\epsilon}^{s+i\epsilon} \frac{dz}{z} \, d(-z) \,. \tag{26}$$

The integration contour in Eq. (26) lies in the region of analyticity of the integrand and encircles the cut of $d(-z)$ on the positive real z axis.

Expressions (25) and (26) can be rewritten in terms of an effective spectral function $\rho^{\rm eff}(\sigma)$

$$d(Q^2) = \frac{1}{\pi} \int_0^\infty d\sigma \frac{\rho^{\rm eff}(\sigma)}{\sigma + Q^2} \tag{27}$$

and [28]

$$r(s) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma} \, \rho^{\rm eff}(\sigma) \,. \tag{28}$$

The effective spectral function $\rho^{\rm eff}(\sigma)$ is determined from the discontinuity of the function $d(Q^2)$ across the cut.

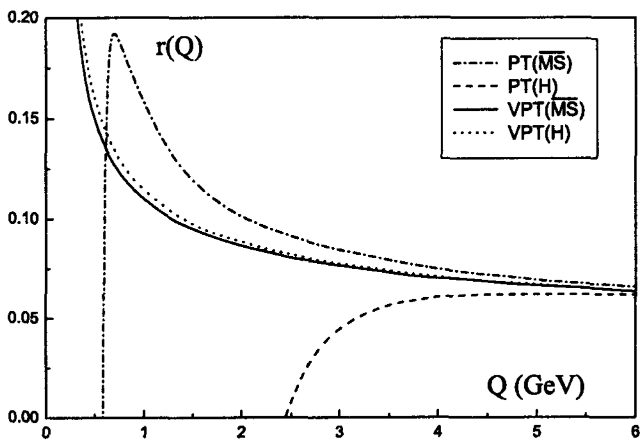

Figure 1: *Plot the QCD corrections $r(Q)$ calculated in the cases of perturbation theory (PT) and the variational perturbation theory (VPT) in two renormalization schemes H and* $\overline{\rm MS}$ *with the same cancellation index* $C_R \simeq 2$.

The QCD contribution to the physical (RS-invariant) quantity $R(s)$ in PT is

$$r = \delta\left[1 + r_1\delta + r_2\delta^2\right], \tag{29}$$

where

$$r_1 = d_1\,, \qquad r_2 = d_2 - \frac{(\beta_0\pi)^2}{48}\,. \tag{30}$$

To discuss the RS-independence, we consider, as in [3], two examples of RS's: the scheme H with the parameters $r_1^H = -3.2$ and $b_2^H = 0$ (the so-called 't Hooft scheme), and the second is the $\overline{\mathrm{MS}}$-scheme corresponding to the parameters $r_1^{\overline{\mathrm{MS}}} = 1.64$ and $b_2^{\overline{\mathrm{MS}}} = 4.47$. From point of view the cancellation index criterion these schemes are close to each other and to the boundary cancellation index $C_H \simeq C_{\overline{\mathrm{MS}}} \simeq C_{PMS} \simeq 2$.

In Fig. 1, we plot the QCD correction $r(Q^2)$ as a function of Q^2 for these two schemes. One can see a stable behavior for the whole interval of energies being practically scheme-independent.

5 Conclusions

We have found that the QCD contribution to the Drell ratio calculated within the VPT method turned out to be practically scheme-independent in a wide class of RS for the whole energy interval. In the variational perturbation theory, therefore, the three-loop level reached presently for a number of physical processes is practically invariant with respect to the choice of the renormalization prescription.

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