Relativistic Inverse Problem for the Superposition of a Non-Local Separable Interaction and a Local Quasipotential not Admiting Bound States

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Abstract

Within the relativistic quasipotential approach to quantum field theory, a relativistic inverse problem is considered for the superposition of a non-local separable quasipotential and a local one that simulates the interaction between two relativistic spinless particles of unequal masses. Besides, the local part of total interaction is supposed to be known and that it does not admit bound states. Then a non-local separable part of total interaction can be reconstructed on the basis of the additional phase-shift and bound-state energies.

The inverse problem in principle was solved within non-relativistic theory by Gelfand and Levitan [1], Marchenko [2], and Krein [3]. In the majority of studies, however, the problem of reconstructing interaction is formulated on the basis of the non-relativistic Schrödinger equation [4–7]. The most complete survey of this theory was given in the monographs of Chadan and Sabatier [8] and Zakhariev and Suzko [9]. Therefore, the problem of reconstructing interaction for essentially relativistic systems, in particular, within the relativistic quasipotential approach [10], is yet remained important.

In the present studies, we consider the problem of reconstructing a non-local separable part of total quasipotential that simulates the interaction between two relativistic spinless particles of unequal masses ($m_1 \neq m_2$)

by means of the additional phase-shift and bound-state energies. The given consideration holds within the relativistic quasipotential approach to quantum field theory proposed in [11]. Besides, the local part $W(r)$ of total interaction is supposed to be known and that it does not admit bound-states. The given approach is based on the expression that was found by the present author for the additional phase-shift $\delta_l^V(\chi')$ and which has the form (we use the system of units where $\hbar = c = 1$)

$$\tan\delta_l^V(\chi') = -\frac{\pi}{2}\sinh^{-1}\chi' A_l(\chi')\left[1 + \mathrm{P}\frac{1}{2}\int_0^\infty d\chi \frac{A_l(\chi)}{\cosh\chi - \cosh\chi'}\right]^{-1}, \quad (1)$$

$$A_l(\chi') = \frac{2}{\pi}\varepsilon_l Q_l^2(\coth\chi')\left|\tilde{V}_l(\chi')/F_l^W(\chi')\right|^2, \quad \varepsilon_l = \pm 1. \quad (2)$$

Here, P means the principal value, $Q_l(z)$ is a Legendre function of the second kind, and $F_l^W(\chi')$ is the Jost function of the local quasipotential $W(r)$ connecting with its the phase-shift $\delta_l^W(\chi')$ by the expression

$$F_l^W(\chi') = \left|F_l^W(\chi')\right|\exp\left[-i\delta_l^W(\chi')\right].$$

Note, that the quantity χ' is the rapidity and it is defined via the relation

$$E_{q'} = \sqrt{m'^2 + \vec{q}'^2} = m'\cosh\chi', \quad m' = \sqrt{m_1 m_2}.$$

In order to find the quasipotential $V_l(r)$ on the basis of the additional phase-shift $\delta_l^V(\chi')$, it is necessary to solve the integral equation (1) concerning of the function $A_l(\chi')$. After that, the function $\tilde{V}_l(\chi')$ is determined from Eq.(2) using the integral Hilbert transformation. The quasipotential $V_l(r)$ is then reconstructed by performing the relativistic Hankel transformation

$$V_l(r) = \int_1^\infty d\rho_l(\cosh\chi)\tilde{V}_l(\chi)\varphi_l(r,\chi), \quad (3)$$

$$d\rho_l(\cosh\chi)/d(\cosh\chi) = (2/\pi)\sinh^{-1}\chi\left|Q_l(\coth\chi)/F_l^W(\chi)\right|^2, \quad (4)$$

where the function $\varphi_l(r,\chi)$ satisfying the boundary condition

$$\lim_{r\to 0}\frac{(-1)^{l+1}\Gamma(l+1)}{(-r)^{l+1}}\varphi_l(r,\chi) = 1,$$

is the regular solution of the finite-difference quasipotential equation with the local quasipotential $W(r)$, and $d\rho_l(\cosh\chi)/d(\cosh\chi)$ is its the spectral density. Here, $\Gamma(z)$ is a gamma function, and

$$(-r)^{(\lambda)} = i^{\lambda}\Gamma(ir+\lambda)/\Gamma(ir).$$

Note, that the integral transformation in (3) is the generalization on the relativistic integral Hankel transformation introduced in [12], and reduces to it when $W(r) \equiv 0$.

We assume that the additional phase-shift $\delta_l^V(\chi')$ in expression (1) is a function continuous in the sense of Hölder with a positive index and that, for $\chi' \to \infty$, it behaves as

$$\delta_l^V(\chi') = O\left((\chi')^{-\gamma}\right), \quad l \geq 0, \quad \gamma > 1. \tag{5}$$

These constraints are necessary and sufficient for the quasipotential to satisfy the condition

$$rV_l(r) \in L_1(0,\infty), \tag{6}$$

which ensures the uniqueness of the inverse-problem solution.

At first, we consider the inverse problem at $\varepsilon_l = +1$. In this case, the additional phase-shift $\delta_l^V(\chi')$ at $\chi' \to +\infty$ must be a nagetive quantity of small magnitude. The energies values then

$$E_{fk} = \cosh\chi_{fk} \geq 1, \quad k = 0,1,\ldots,\nu_l - 1, \tag{7}$$

corresponding of the "spurious" bound-states [4], must satisfy the conditions

$$\delta_l^V(\chi_{fk}) = \pi k, \quad k = 0,1,\ldots,\nu_l - 1. \tag{8}$$

In this case, the Levinson theorem has the form

$$\delta_l^V(0) - \delta_l^V(\infty) = \delta_l^V(0) = \pi\nu_l, \tag{9}$$

$$\delta_l^W(0) - \delta_l^W(\infty) = \delta_l^W(0) = 0.$$

The integral equation (1) can be reduced to the form

$$\psi_l(x) = 1 + \frac{1}{\pi}\int_1^{\infty} dt \frac{\psi_l(t)h_l^*(t)}{t - x - io}, \tag{10}$$

where we introduced the following notation:

$$\begin{aligned}
\psi_l(x) &= A_l(\operatorname{arc}\cosh x) g_l^{-1}(x)\left[1 + i(\pi/2) g_l(x)(x^2-1)^{-1/2}\right], \qquad (11)\\
g_l(x) &= -(2/\pi)(x^2-1)^{1/2}\tan\Delta_l^V(x),\\
\Delta_l^V(x) &= \delta_l^V(\operatorname{arc}\cosh x), \quad x = \cosh\chi',\\
h_l(x) &= (\pi/2) g_l(x)(x^2-1)^{-1/2}\left[1 - i(\pi/2) g_l(x)(x^2-1)^{-1/2}\right]^{-1} =\\
&= -\sin\Delta_l^V(x)\exp\left[-i\Delta_l^V(x)\right].
\end{aligned}$$

Let us consider the function

$$H_l(z) = 1 + \frac{1}{\pi}\int_1^\infty dt \frac{\psi_l(t) h_l^*(t)}{t-z}. \qquad (12)$$

It is obvious that the function $H_l(z)$ is analytic in the complex plane of the variable z with the cut from 1 to $+\infty$ along the real axis and the relation

$$\lim_{|z|\to\infty} H_l(z) = 1 \qquad (13)$$

holds in all directions, if the function $\psi_l(x)$ is continuous in the sense of Hölder and if the integral in Eq.(12) converges. Hence, a solution to the integral equation (10) can be represented as

$$\psi_l(x) = H_l(x_+) = \lim_{\eta\to+0} H_l(x+i\eta), \quad 1 \le x \le \infty. \qquad (14)$$

By substituting the solution in (14) into the expression for the discontinuity suffered by the function $H_l(z)$ upon traversing the cut,

$$H_l(x_+) - H_l(x_-) = -2i\sin\Delta_l^V(x)\exp\left[i\Delta_l^V(x)\right]\psi_l(x), \qquad (15)$$

we arrive at the homogeneous Riemann–Hilbert equation for the function $H_l(z)$:

$$H_l(x_+)\exp\left[2i\Delta_l^V(x)\right] - H_l(x_-) = 0, \quad 1 \le x \le \infty. \qquad (16)$$

A particular solution satisfying Eq.(16) and the condition in (13) has the form

$$\tilde{H}_l(z) = \exp\left[\omega_l(z)\right], \qquad (17)$$

where

$$\omega_l(z) = -\frac{1}{\pi}\int_1^\infty dt\frac{\Delta_l^V(t)}{t-z}. \qquad (18)$$

We also have the relation

$$\lim_{|z|\to\infty} \omega_l(z) = 0,$$

which holds in all directions, as follows from the assumptions on the behaviour of the additional phase-shift and from the conditions in (5). Besides, its behaviour is given by

$$\omega_l(z) = \frac{1}{\pi}\Delta_l^V(1)\ln|1-z| + \Omega_l(z), \quad z \to 1. \tag{19}$$

Here, the function $\Omega_l(z)$ is finite for $z \to 1$, while $\Delta_l^V(1) = \delta_l^V(0) = \pi\nu_l$ in according with the Levinson theorem in(9). Therefore, the function $\tilde{H}_l(z)$ has a zero of order ν_l at the point $z = 1$. This means that, according to (14), (17) and (18), the particular solution to the nonhomogeneous integral equation (10) has the form

$$\tilde{\psi}_l(x) = \exp\left[\alpha_l(x) - i\Delta_l^V(x)\right], \tag{20}$$

where

$$\alpha_l(x) = -\frac{1}{\pi}\mathrm{P}\int_1^\infty dt \frac{\Delta_l^V(t)}{t-x}. \tag{21}$$

Note, that the function given by (20) is regular at $x = 1$ (it has a zero of order ν_l at this point), is continuous in the sense of Hölder with the same index as the additional phase-shift, and is bounded for $x \to +\infty$. All this is concides with the a priori assumptions on its properties. Finally, this function is a particular solution to Eq.(10) since, according to the Cauchy theorem, we have

$$\lim_{\substack{R\to+\infty \\ \eta\to+\infty}} \frac{1}{2\pi i}\int_{\Gamma^+} dz \frac{\tilde{H}_l(z)}{z-x-i\eta} = 1 + \frac{1}{\pi}\int_1^\infty dt \frac{\tilde{\psi}_l(t)h_l^*(t)}{t-x-io} = \tilde{H}_l(x_+) = \tilde{\psi}_l(x).$$

Here, Γ^+ is the closed contour consisting of a circle C_R^+ of radius R and center at the point $z = 0$, two banks of the cut from 1 to R that it goes in opposite directions along these banks, and a circle C_η^- having a radius η and a center at the point $z = 1$. The contribution of the integral along the circle C_η^- tends to zero for $\eta \to +0$ since the function $\tilde{H}_l(z)$ has a zero of order ν_l at the point $z = 1$.

A general solution to the homogeneous equation

$$\psi_{lo}(x) = \frac{1}{\pi}\int_1^\infty dt \frac{\psi_{lo}(t)h_l^*(t)}{t-x-io} \tag{22}$$

we will look for in the form

$$\psi_{lo}(x) = H_{lo}(x_+) = \exp[\omega_l(x_+)]\sum_{k=1}^{m}\frac{A_k}{(x-1)^k}, \tag{23}$$

where the function

$$H_{lo}(z) = \frac{1}{\pi}\int_1^\infty dt \frac{\psi_{lo}(t)h_l^*(t)}{t-z}, \tag{24}$$

satisfying the homogeneous Riemann-Hilbert equation in (16), is analitic in the complex plane of the variable z with the cut from 1 to $+\infty$ along the real axis, in additional to the relation

$$\lim_{|z|\to\infty} H_{lo}(z) = 0 \tag{25}$$

holds in all directions. Substituting (23) into (16) and requiring that the function in (24) be finite at $z = 1$, we obtain $m = \nu_l$. Hence, we have

$$\psi_{lo}(x) = H_{lo}(x_+) = \exp\left[\alpha_l(x) - i\Delta_l^V(x)\right]\sum_{k=1}^{\nu_l}\frac{A_k}{(x-1)^k}. \tag{26}$$

It is obvious that, as in the case of a particular solution, the function in (26) satisfies Eq.(22) and possesses all the required properties.

Thus, a general solution to the integral equation (10) has the form

$$\psi_l(x) = \tilde{\psi}_l(x) + \psi_{lo}(x) = \exp\left[\alpha_l(x) - i\Delta_l^V(x)\right]\left[1 + \sum_{k=1}^{\nu_l}\frac{A_k}{(x-1)^k}\right]. \tag{27}$$

At last, by using the notation in (11) and transforming the sum as a product, we can recast the solution in (27) into the form

$$A_l(\chi') = -\frac{2}{\pi}\sinh\chi'\sin\delta_l^V(\chi')\exp\left[\alpha_l(\cosh\chi')\right]\prod_{k=0}^{\nu_l-1}\left[1 + a_k/(\cosh\chi'-1)\right], \tag{28}$$

where

$$\alpha_l(\cosh\chi') = -\frac{1}{\pi} \mathrm{P} \int_0^\infty d\chi \frac{\sinh\chi\,\delta_l^V(\chi)}{\cosh\chi - \cosh\chi'}. \tag{29}$$

In order to determine the constants a_k in (28), we note that, in accordance with the definition in (2), the function $A_l(\chi')$ is of fixed sign at all values of χ', and so far as $\varepsilon_l = +1$, that it must be positive. At the same time, the additional phase-shift satisfies the conditions in (8) at the "spurious" bound-states energies in (7). Hence, the function $A_l(\chi')$ retains a plus sign, provided that

$$a_k = 1 - \cosh\chi_{fk}, \quad k = 0, 1, \ldots, \nu_l - 1.$$

Instead of (28), we will then have

$$A_l(\chi') = -\frac{2}{\pi}\sinh\chi' \sin\delta_l^V(\chi') \exp\left[\alpha_l(\cosh\chi')\right] \prod_{k=0}^{\nu_l-1} \left[1 - \left(\frac{\sinh(\chi_{fk}/2)}{\sinh(\chi'/2)}\right)^2\right]. \tag{30}$$

Thus, the solution in (30) is completely determined by the additional phase-shift so far as χ_{fk} is also determined by its the behaviour. Moreover, it follows from expressions (29) and (30) that the function $A_l(\chi')$ is continuous in the sense of Hölder and that, for $\chi' \to +\infty$, it behaves as

$$\cosh\chi' |\chi'|^{-\gamma}, \quad \gamma > 1,$$

provided that the additional phase-shift satisfies condition (5). This in my turn implies that the quasipotential $V_l(r)$ satisfies condition (6).

The case where $\varepsilon_l = -1$ and where there are ν_l the "spurious" bound-states at energies in (7) satisfying the conditions (8) ($k = 1, 2, \ldots, \nu_l$), and one true bound-state whose energy lie in the range

$$0 \le E_t = \cosh\chi_t < 1, \quad \chi_t = i\kappa_t, \quad 0 < \kappa_t \le \pi/2, \tag{31}$$

is considered in the same way. Besides, by the Levinson theorem, we have

$$\delta_l^V(0) - \delta_l^V(\infty) = \delta_l^V(0) = \pi(\nu_l + 1).$$

In accordance with expression (19), the function $\tilde{H}_l(z)$ therefore has a zero of order $(\nu_l + 1)$ at $z = 1$. Further following in the same way as for the case of $\varepsilon_l = +1$ and considering that the function $A_l(\chi')$ must now retain a minus sign at all values of χ', so far as $\varepsilon_l = -1$, we obtain

$$A_l(\chi') = -\frac{2}{\pi} \sinh\chi' \sin\delta_l^V(\chi') \exp\left[\alpha_l(\cosh\chi')\right] \times \left[1 + \left(\frac{\sin(\kappa_t/2)}{\sinh(\chi'/2)}\right)^2\right] \prod_{k=1}^{\nu_l} \left[1 - \left(\frac{\sinh(\chi_{fk}/2)}{\sinh(\chi'/2)}\right)^2\right]. \quad (32)$$

Thus, the function $A_l(\chi')$ is completely determined by the additional phase-shift and true bound-state energy too, and its sign is contrary to the sign of the additional phase-shift for $\chi' \to +\infty$.

In order to reconstruct the quasipotential $V_l(r)$ by means of the transformation in (3), we can introduce the function

$$\hat{V}_l(\sinh(\chi'/2)) = \frac{\sinh(\chi'/2) + i\sin(\kappa_t/2)}{\sinh(\chi'/2) - i\sin(\kappa_t/2)} \times \left|Q_l(\coth\chi')/F_l^W(\chi')\right|^2 \left[\tilde{V}_l^{(-)}(\sinh(\chi'/2))\right]^2, \quad (33)$$

where

$$\left|\tilde{V}_l^{(-)}(\sinh(\chi'/2))\right| = \left|\tilde{V}_l(\chi')\right|, \quad \mathrm{Re}\tilde{V}_l^{(-)}(\sinh(\chi'/2)) = \mathrm{Re}\tilde{V}_l(\chi'), \quad (34)$$

$$\arg\tilde{V}_l^{(-)}(-\sinh(\chi'/2)) = -\arg\tilde{V}_l^{(-)}(\sinh(\chi'/2)), \quad \arg\tilde{V}_l(-\chi') = \arg\tilde{V}_l(\chi').$$

The function $\hat{V}_l(\sinh(\chi'/2))$ is analitic in the region $0 < \mathrm{Im}\chi' \leq \pi/2$, it is continuous for $0 \leq \mathrm{Im}\chi' \leq \pi/2$ and satisfies the condition

$$\hat{V}_l(\sinh(\chi'/2)) = O(\sinh^2(\chi'/2)), \quad |\chi'| \to \infty, \quad 0 \leq \mathrm{Im}\chi' \leq \pi/2, \quad (35)$$

provided that the condition in (6) is carried out. Besides, the function $\hat{V}_l(\sinh(\chi'/2))$ vanishes nowhere for $0 < \mathrm{Im}\chi' \leq \pi/2$. Hence, the function $\ln\hat{V}_l(\sinh(\chi'/2))$ is analitic in the region $0 < \mathrm{Im}\chi' \leq \pi/2$ and it behaves as $\ln\sinh^2(\chi'/2)$ for $|\chi'| \to \infty$ because of the estimate in (35). Therefore, we can apply the integral Hilbert transformation to the real and the imaginary parts of the function $\ln\hat{V}_l(\sinh(\chi'/2))$. For the real values of χ', we then obtain

$$\mathrm{Im}\ln\hat{V}_l(\sinh(\chi'/2)) = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} d(\sinh(\chi/2)) \frac{\mathrm{Re}\ln\hat{V}_l(\sinh(\chi/2))}{\sinh(\chi/2) - \sinh(\chi'/2)} = \quad (36)$$

$$= -\frac{2\sinh(\chi'/2)}{\pi}\mathrm{P}\int\limits_0^\infty d\chi\frac{\cosh(\chi/2)\ln[\pi\varepsilon_l A_l(\chi)/2]}{\cosh\chi - \cosh\chi'}.$$

Combining the expressions (33) and (36), we now obtain the formula

$$\left|\frac{Q_l(\coth\chi')}{F_l^W(\chi')}\right| \tilde{V}_l^{(-)}(\sinh\chi'/2) = \sqrt{\pi\varepsilon_l A_l(\chi')/2}\times$$

$$\times\exp\left\{-i\arctan\left(\frac{\sin(\kappa_t/2)}{\sinh(\chi'/2)}\right)\right.$$

$$\left.-\frac{i\sinh(\chi'/2)}{\pi}\mathrm{P}\int_0^\infty d\chi\frac{\cosh(\chi/2)\ln[\pi\varepsilon_l A_l(\chi)/2]}{\cosh\chi-\cosh\chi'}\right\}. \quad (37)$$

At last,from expressions (34) and (37), it follows that

$$\left|\frac{Q_l(\coth\chi')}{F_l^W(\chi')}\right| \tilde{V}_l(\chi') = \sqrt{\pi\varepsilon_l A_l(\chi')/2}$$

$$\exp\left\{-i\mathrm{sgn}\chi'\left[\arctan\left(\frac{\sin(\kappa_t/2)}{\sinh(\chi'/2)}\right)+\right.\right.$$

$$\left.\left.\frac{\sinh(\chi'/2)}{\pi}\mathrm{P}\int_0^\infty d\chi\frac{\cosh(\chi/2)\ln[\pi\varepsilon_l A_l(\chi)/2]}{\cosh\chi-\cosh\chi'}\right]\right\}, \quad (38)$$

which is valid for the real values of χ'.

Thus, a solution to the inverse problem exists and is completely determined as the function $A_l(\chi')$ is found by the additional phase-shift and true bound-state energy for $l \geq 0$.

To summarize, we note that the method proposed here to reconstruct a non-local separable part of total quasipotential simulating the interaction between two relativistic spinless particles of unequal masses actually reduces to a one-body problem. This is thanks to the possibility of representing, within the relativistic quasipotential approach to quantum field theory, the total c.m. energy of two relativistic particles of unequal masses as an expression proportional to the energy of an effective relativistic particle of mass m'.

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References

[1] I.M.Gel'fand and B.M.Levitan, Dokl.Akad.Nauk SSSR **77**, 557 (1951); Izv.Akad.Nauk SSSR, Ser.Mat. **15**, 309 (1951).

[2] V.A.Marchenko, Dokl.Akad.Nauk SSSR **104**, 695 (1955).

[3] M.G. Kreĭn, Dokl.Akad.Nauk SSSR **76**, 21 (1951).

[4] K.Chadan, Nuovo Cimento **XLVII** A, 510 (1967).

[5] M.Bolsterli and J.MacKenzie, Physics **2**, 141 (1965).

[6] F.Tabakin, Phys.Rev. **177**, 1443 (1969).

[7] R.L.Mills and J.F.Reading, J.Math.Phys. **10**, 321 (1969).

[8] K.Chadan and P.C.Sabatier. *Inverse Problems in Quantum Scattering Theory* (Springer-Verlag, New York, 1977; Mir, Moscow 1980).

[9] B.N.Zakhariev and A.A.Suzko, *Direct and Inverse Problems: Potentials in Quantum Scattering* (Energoatomizdat, Moscow, 1985; Springer-Verlag, Berlin, 1990).

[10] A.A.Logunov and A.N.Tavkhelidze, Nuovo Cimento **29**, 380 (1963).

[11] V.G.Kadyshevsky, Nucl. Phys. B **6**, 125 (1968).

[12] Yu.D.Chernichenko, Yad.Fiz. **63**, 2068 (2000).